Targil 9 (following Alexey's story, about catastrophes and linear variations)

1. Given a polynomial of degree 3 with real coefficients: $x^3 + ax + b$.
Prove that it has 3 distinct real roots iff $(a/3)^3+(b/2)^2$ is negative;
and that it has only one real root of multiplicity one iff $(a/3)^3+(b/2)^2$ is positive.

First solution. The sum of roots is 0, the sum of product of pairs is a, the product of al 3 is $-b$. Consider different cases:
First case: we have 3 real roots, m , n , and $-m-n$ (cause the sum is equal 0), and WLOG we can assume that m, n are both negative or both positive (if we have a zero root, the statement is obvious).
Then $a = -m(m+n) - n(m+n) + mn = -(m^2 + mn + n^2)$
$b = mn(m+n)$
Hence it remains to verify that

$$-\left(\frac{m^2+mn+n^2}{3}\right)^3+\left(\frac{mn(m+n)}{2}\right)^2<0$$

$$\left(\frac{mn(m+n)}{2}\right)^2<\left(\frac{m^2+mn+n^2}{3}\right)^3$$

Divide by m^3n^3 and denote $x = m/n$, and it is positive.

$$\left(\frac{x^{1/2}+x^{-1/2}}{2}\right)^2<\left(\frac{x+1+x^{-1}}{3}\right)^3$$

Denote $y = x^{1/2} + x^{-1/2}$. Then $y > 2$.

Then $x+1+x^{-1} = y^2 - 1$ and our inequality becomes:

$$\frac{y^2}{4}<\frac{\left(y^2-1\right)^3}{27}$$

Denote $z = y^2 - 1$, then $z > 3$, and it remains to prove:

$$\frac{27}{4}(z+1)<z^3$$

For $z = 3$ we get equality, but the right hand side climbs faster, because its derivative is greater: $\frac{27}{4}<3z^2$, hence the inequality holds for $z > 3$.

Second case. We have a double root, so the roots are k, k, and $-2k$.
Then it is easy to compute $(a/3)^3+(b/2)^2 = 0$.

Third case. We have two complex conjugated roots $m+in$, $m-in$, and a real root $-2m$.
Then $b = -2m(m^2+n^2)$ and $a = m^2+n^2 - 4m^2 = n^2 - 3m^2$. So, it remains to prove that in this case

$$\left(\frac{n^2-3m^2}{3}\right)^3+\left(\frac{-2m\left(m^2+n^2\right)}{2}\right)^2>0$$

$$\left(m\left(m^2+n^2\right)\right)^2>\left(m^2-\frac{n^2}{3}\right)^3$$

$$m^6+2m^4n^2+m^2n^4>m^6-m^4n^2+\frac{m^2n^4}{3}-\frac{n^6}{27}$$

$$3m^4n^2+\frac{2}{3}m^2n^4+\frac{n^6}{27}>0$$

And that is obvious.

That was a straightforward and messy solution.
Let us see a nicer, catastrophic one.

Solution 2. Consider the discriminant of the polynomial – the resultant of it with its derivative. (If you forgot or don't know about resultants, read the solution of problem 2 from targil 1)

$$\begin{vmatrix} 1 & 0 & a & b & 0 \\ 0 & 1 & 0 & a & b \\ 3 & 0 & a & 0 & 0 \\ 0 & 3 & 0 & a & 0 \\ 0 & 0 & 3 & 0 & a \end{vmatrix} = \begin{vmatrix} 1 & 0 & a & b & 0 \\ 0 & 1 & 0 & a & b \\ 0 & 0 & -2a & -3b & 0 \\ 0 & 0 & 0 & -2a & -3b \\ 0 & 0 & 3 & 0 & a \end{vmatrix} = \begin{vmatrix} -2a & -3b & 0 \\ 0 & -2a & -3b \\ b & 0 & a \end{vmatrix} = 4a^3+27b^3$$

Hence $(a/3)^3+(b/2)^2 = 0$ iff there is a double root. The line $a=3\sqrt[3]{(b/2)^2}$ divides the a,b plane into two parts, one is $a>3\sqrt[3]{(b/2)^2}$ and another is $a<3\sqrt[3]{(b/2)^2}$.
If we move a couple of complex conjugated roots and the real root along the plane, without creating double root, we can continuously transform the polynomial with just one real root into a polynomial with just one real root, and the sign of the discriminant won't change since the discriminant won't go through 0.
Similarly, if we move all three real roots, keeping their order, we shall keep the sign of discriminant. Hence, each case, of 1 real root and of 3 real roots, correspond to a connected component of the plane with the zeroes of discriminant cut out.
It remains to verify, which of the 2 connected components (or alternatively, which sign of discriminant) corresponds to 1 real root and which to 3 real roots. To check it, substitute an example. For instance a polynomial with roots 1, –1, 0, which is x^3-x.
Or a polynomial with 1 real root 0, and two complex i, $-i$, whichi is x^3+x.

2. In Moscow they have 9 sky-scrapers (assume Moscow is a plane, and sky-scrapers are points). A tourist, that stands at a certain point and looks around (counterclockwise), will

see them in a certain cyclical order, at least if he doesn't stand on a line connecting two sky-scrapers. There might be 8! different cyclic orders.
(a) is it possible that each order will appear at some point?
(b) for which minimal number of buildings, will it be possible to see the buildings in every cyclic order?

Solution. (a) Let us count the number of areas we get, when we draw all the lines going through the couples of all the skyscraper points. This number will give us an upper bound for the possible number of observable cyclic orderings of the buildings, since all the points in the same part give you the same ordering, since no changes in the order of the buildings can happen if you don't cross any line.

For N skyscrapers we get $\binom{N}{2}=\frac{N(N-1)}{2}$ couples/lines. Each line is intersected by many lines at the 2 points that define it, and it may also be intersected at all lines defined by other N–2 points, if there are no trapezoids (no parallel lines). Also, we can complete (compactify) the plane by the infinite point, which is contained in all lines, and then we shall have a map (meaning countries and borders) on the sphere, and we shall be able two use Euler's formula. So, on each line we shall have $3+\binom{N-2}{2}$ points and the same number of edges (at most, if there are no trapezoids or coincidences), so total number of edges is at most $E=\binom{N}{2}\cdot\left(3+\binom{N-2}{2}\right)$.

Each trapezoid or coincidence reduces the number of faces (it is easy to see, that if you stir the points a little bit to avoid coincidence, you have all faces you had previously and usually more, so if we count the number of faces when there are no coincidences, we shall get the possible maximum.

The total number of vertices is: N original points, 1 at infinity, and number of pairs of couples (since each two lines give an intersection $\frac{N(N-1)(N-2)(N-3)}{8}$.

Since F – E + V = 2, we get F = 2 + E – V, hence maximal F is

$$2+\binom{N}{2}\cdot\left(3+\binom{N-2}{2}\right)-\left(N+1+\frac{N(N-1)(N-2)(N-3)}{8}\right)=$$

$$=1-N+\binom{N}{2}\left(3+\frac{(N-2)(N-3)}{2}\left(1-\frac{1}{2}\right)\right)=$$

$$=1-N+3\frac{N(N-1)}{2}+\frac{N(N-1)(N-2)(N-3)}{8}$$

Of course, it would be easier to substitute the numbers from the beginning, but we want to prove a stronger claim. Assume we have N=6 buildings in Moscow, then the plane is divided into only

$1-6+3\frac{6\cdot 5}{2}+\frac{6\cdot 5\cdot 4\cdot 3}{8}=1-6+45+45=85$ parts, and there are 5! = 120 possible cyclic orderings for 6 buildings, which is considerably more, so even if we choose a subset of 6 buildings, we won't be able to observe them in arbitrary order.

Now let us substitute N=5

$1-5+3\frac{5\cdot 4}{2}+\frac{5\cdot 4\cdot 3\cdot 2}{8}=1-5+30+15=41.$

It is more then 5! = 120, so we cannot extend the statement by this method for 5 buildings.
Of course, it cannot be considered a proof that for 5 we have a configuration of buildings either, since different parts of the plane, even those separated by lines might correspond to the same cyclic order.

(b) We have found some upper bound, lets start with lower. If we have 3 points in "generic situation" they form a triangle, and there are two possible cyclic orders, one is observed from each internal point, and another from almost every external point (w. r. t. that triangle), so 3 is still possible.
4 points in generic position can form 2 kinds of configuration – either a convex quadrilateral or a triangle and a point inside. In both cases, it is not too hard to check you have all orderings.
As for 5 points, I had a really nice proof that some orderings are not achieved when I gave this problem, but as I found out when I started typing it down, it was wrong. Right now I don't know neither the solution, nor the answer.

3. A convex body C is contained by the unit cube. Projection of C to every face of the cube covers it completely. What is the minimal possible volume of C?

Solution. Every edge meets this convex body, otherwise projection to the orthogonal face wouldn't cover one of the corners. Conversely, each edge meets the body, than each projection covers 4 corners, then, since it is convex, it covers the face. So we can take the condition of taking a point from each of 12 edges, and forget the original condition. After we choose 12 points, one on each edge, we should take their convex hull, and the volume of that body should be minimized y a smart choice of points on the edges. So, we have reduced the problem with infinite number of parameters to a problem with 12 parameters. Instead of computing/minimizing the volume of the convex hull, it is easier compute/maximize the volume of the complement. It consists of the "corners" – right-angled tetrahedrons, adjacent to each vertex of the cube. 4 vertices of such a tetrahedron are the vertex of the cube, and 3 points we have chosen on the edges, attached to that cube vertex.
Let us try moving one of those 12 points on edges, and look how it influences the volume of the complement. When we try to move it along the edge, only 2 corner tetrahedrons change, and their both volumes depend linearly on the position of that point. Hence, the maximum (and the minimum) will occur only when the point is on one of extreme positions. So, we may assume that in the best situation all points we choose on each of 12

edges are vertexes. Hence, the volume of each corner tetrahedron is either 0 (if these point was chosen as one of those 12 points) or 1/6 (if all neighbor vertices were chosen). Divide all vertices into 4 subsets of 2, members of the same subgroup are connected by vertical edges. Only one of the members of each subgroup can invest 1/6 to the volume of the complement, the other will give 0. So we can have at most 4/6 = 2/3 of volume in the complement, so the volume of the original body is at least 1/3.
From the above, it is easy to guess the construction for the 1/3.
Color the vertices of the cube in two colors, black and white, so that neighbors will be of different colors. Now take the convex hull of the white vertices.

4. A rectangle is divided into disjoint union of the finite number of squares. Prove, that the aspect ratio (width / height) of the rectangle is a rational.

Solution. Consider the certain sub-division of the certain rectangle into squares. Denote x_i the sides of the squares. The subdivision defines certain linear equations over the numbers x_i, such as: if a certain interval in the picture is presented as sum of sides of different subsets of squares, then those sums should be equal. Two more equations clame that sum of all sides of squares touching the lower rectangle's border is w, and sum of two rectangle's squares touching the right rectangle's border is h.
All coefficients in that system of equations are rational, except for w and h. After applying scaling we shall assume $h = 1$, and then w will be the aspect ratio.
The configuration of squares solves the problem for given w if and only if the system of linear equations has a solution in nonnegative real numbers. When we apply Gauss method to solve the system of all equations except the one containing w, we shall get either a single rational solutions (because coefficients are rational), or an infinite family of solutions, which depends linearly (with rational coefficients) upon a finite number of parameters. A solution we get from Gauss methods is limited by several inequalities, corresponding to non-negativity of all x_i. So, if we have an infinite family of solution (and that is the only way to get irrational w) then w moves in certain limits, between two rational limiting values, a and b.
In such a case, we might write $x_i = k_i t + m_i$, and for every t we shall get the same configuration of squares, but of different sizes. Say at $t = 0$ we shall get rectangle of width a, at $t = 1$ we shall get a rectangle of width b, and at some intermediate value we shall get width w. Sides of all squares change as a linear function of t. So their areas are quadratic functions, with corns up (all smiling), unless they are unchanged. But height is constantly 1, and w changes linearly, hence the area changes linearly. That's a contradiction.
Hence, there are no configurations which give infinite families of solutions, only those, that give a single rational solution.

5. A square matrix is called ***bi-stochastic*** if all its numbers are nonnegative and sum of numbers in each column and in each row is one.
Prove that any bi-stochastic matrix is a linear combination of permutation matrix, with positive coefficients.

Solution. When talking about elements of bi-stochastic numbers, all numbers in the open interval (0, 1) will be called ***fractional***, to distinguish them from 0 and 1 which will be called ***extreme***.
We shall prove the statement by induction over the number of fractional matrix entries. If we have 0 fractional elements in the matrix, all elements are 0s or 1s, it is easy to see that the matrix is a permutation matrix, and that is the base of induction.

Consider a matrix with K fractional elements. In the same column with any fractional element we obviously have a fractional element. In the same row with any fractional element we also have a fractional element.
Start with arbitrary fractional element of the matrix and build a sequence of fractional elements, passing with each odd time to a different fractional element of the same row and in each even time to a different fractional element in the same column. At a certain moment, You will have to repeat an element which occurred before in that sequence. We can assume that the repeated element will be repeated after even number of moves (if for example, we want to make a vertical move from element Q to A, then either A was followed by a horizontal move, or A was followed by a vertical move to B, but then we can move directly from F to B). Hence, from this sequence, we can choose a cycle of fractional, each even element connected by vertical move to the next and by horizontal move to the previous.
If we add the same number x to all odd numbers in that cycle and subtract it from all even numbers of that cycle, then the sum in each column and the sum in each row is preserved. Choose maximal possible x and minimal possible x so that all elements of the matrix will be still in [0,1]. We shall get two matrices, C and D, both bi-stochastic, which both have less fractional elements than the original matrix. Hence, by induction, C and D are positive linear combinations of permutations.
It is obvious that the original matrix is positive linear combination of C and D.