Targil 7.

1. Consider Rubik's cube (קוביה הונגרית). Is it possible to find a certain sequence of moves, such that you can solve the cube from any situation, if you repeat that specific combination of moves sufficiently many times?

Solution. If we would, the group of rotations of Rubik's cube would be cyclic. Then, it would be commutative. It isn't. Rotations of two adjacent faces don't commute.

2. Consider a table:

1	2	3	4	5	6	7
7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1

You are allowed to flip each two rows and to flip each two columns. How many different tables can You get by a sequence of such steps?

Answer. $7!\cdot 6!$

First solution. Consider the stabilizer of this table (subgroup of operations that keep it). It consists of at least 7 elements (any cyclic rotation of rows with the same rotation of columns). The set of possible tables is the (right) factor over the stabilizer.
We have $7!^2$ elements in group, so no more than $7!^2/7$ possible tables. But we can make any of 7! Permutation in upper row, and 6! permutations of the elements of left column disregarding the upper-left corner, so at least 7!6! different tables.

Second solution. For each 4 cells forming a rectangle ABCD, sides parallel to edges of the table, it is easy to proof that we have A+C = B+D (mod 7) for the values of the cells. This property (a) holds when we perform our operations (b) allows to reconstruct the table by first row and left column only. It is easy to that we can make 7!6! different situations in the first row and left corner, so…

3**. We play the following game. On the circle, we have some red and blue points (always at least two). We can insert a red point into certain arc, and flip the colors of the ends of that arc (red changed to blue, blue changed to red). Conversely, if we have at least three points, we can erase a red point, simultaneously flipping the colors of its neighbors.
In the beginning of the game, we have only two points on the circle and both are blue. Can we perform a sequence of moves, so that in the end we shall have only two colored points, and both red?

First solution. Let R be a clockwise 120^0-rotation of equilateral (regular) triangle, B its reflection with respect to a certain altitude. Then BRB = RR , BRR = RB , RRB = RB (all those equalities are obviously equivalent, first is obvious – conjugating rotation by

reflection gives you a rotation in opposite direction). Now, take your circle, go clockwise from certain point, and write down R for each red and B for each blue point. You get an element of group S_3 (group of 3-permutations or of triangle symmetries) when you write everything down. The element you get depend on the starting point, but its conjugation class doesn't. From the equalities we wrote, we see that the class of conjugacy won't change under those operations.
So, two red points correspond to RR, which is conjugate to rotation (element of order 3) and BB corresponds to identity, so they can't be transformed one into another.

Second solution. The parity of number of red points is an invariant, but it doesn't distinguish between two situations: it is even in both cases. So, assume we have an even number of red points and red points split the circle into even number of arcs. Count the number of blue points on odd arcs minus number of blue points on even arcs modulu 3. One can show with a routine verification it is an invariant, it is 2 in one case and 0 in another, QED.
Actually it is the same as the first solution, though the reason, why this particular invariant works, is unclear here.

4*. Let G be a finite group. For arbitrary subsets U,V,W of G, denote by N_{UVW} the number of triples (x,y,z) in $U \times V \times W$, for which xyz is the unity.
Suppose that G is partitioned into three sets A, B and C (i.e. sets A,B,C are pairwise disjoint and $G = A \cup B \cup C$).Prove that $N_{ABC} = N_{CBA}$.

Solution. Firstly, it is obvious that $N_{CBA} = N_{BAC}$ since zyx = 1 iff yxz = 1 because they are conjugated: $zyx = z \cdot yxz \cdot z^{-1}$. So, enough to prove $N_{ABC} = N_{BAC}$.
The elements (x,y) in $U \times V$ are of 3 kinds: $(xy)^{-1}$ can be in A, B, or C.
So $N_{UVA}+N_{UVB}+N_{UVC} = |U| \cdot |V|$. But $N_{ABA} = N_{BAA}$, $N_{ABB} = N_{BAB}$ because of conjugation, therefore $N_{ABC} = |A| \cdot |B| - (N_{ABA} + N_{ABB}) = |A| \cdot |B| - (N_{BAA} + N_{BAB}) = N_{BAC} = N_{CBA}$.

5**. Let G be a group of $2^k(2m+1)$ elements and suppose it has an element of order 2^k. Prove that all the elements of odd order (together with the unit element) form a subgroup.

Solution. We shall use induction on k. for k = 0 the statement is obvious.
Consider left action of group on itself. Each element of a group, when you multiply group elements by it, defines a permutation of group elements. Any element of odd order defines only odd cycles, so it defines an even permutation. An element of order 2^k defines 2m+1 cycles of even order (2^k), and that is odd permutation.
Consider a subgroup of those group elements that correspond to even permutations. Those elements contain all elements of odd order, but not all elements in the group.
It is a subgroup of order 2, so we reduced our problem to a problem on a smaller group, which follows from induction assumption.