Problems 6 – polynomials

1. Prove that a polynomial with real coefficient p(x) is nonnegative for all real values iff it is a sum of two squares (of polynomials with real coefficients).

2. A polynomial with real coefficients of 2 variables $p(x,y)$ is always positive. Is it true that it is always bigger then some positive ε?

3. (a) Suppose that a polynomial with integer coefficients is can be decomposed into a product of two polynomials with rational coefficients. Show that it is decomposed into a product of two polynomials with integer coefficients.
(b) Suppose p is a prime number and a polynomial with integer coefficients

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

has the following properties:

a_n is not divisible by p,
$a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are divisible by p,
but a_0 is not divisible by p^2.

In this case the polynomial is not decomposable into the product of two polynomials with integer coefficients.

Definition. A polynomial is called ***irreducible*** if it can't be presented as a product of two polynomials of degree>0. Of course, this definition depends on the field (for example, it can be irreducible over **Q** and split over **R**).

4. Determine whether the following polynomials are irreducible over **Q**:
(a) $2x^3+3x^2+5x+7$
(b) $x^4+19x^2+2x+99$
(c) x^8+x+1
*(d) $x^n+x^{n-1}+\ldots+x^2+x+1$ (here answer depends on n)

5. Proof that if two polynomials of degree less then N have the same values at N different points, then they coincide.

6. Given two polynomials $p(z)$, $q(z)$ with complex coefficients. It is also given that for any complex number z ,

$$p(z) = 0 \text{ iff } q(z) = 0$$
$$p(z) = 1 \text{ iff } q(z) = 1$$

Prove that the polynomials are equal.