Targil 5. Combinatorics again, but now with infinite sets.

1. Show that each sequence $\{a_n\}$ of real numbers has either infinite non-decreasing subsequence or infinite non-increasing subsequence.

Solution. Denote $A_m = \{a_n \mid n > m\}$. If any A_m has maximal element, then building non-increasing subsequence is easy: choose maximal element in A_1, then choose maximal element of all elements after it and so on, each time choose maximal element of all elements that have bigger index then all chosen ones, and we have non-increasing sequence.
If some A_k has no maximal element, then for any $m > k$, A_m has no maximal element either. Then for any element there is a bigger element with bigger index, so we can choose a strictly increasing subsequence.

2. Consider a set of distinct points in space $\{(x_i, y_i, z_i)\}$ such that all their coordinates are natural (positive integers). (Not all points of that kind, just some of them.)
Point (x_i, y_i, z_i) of this set is called minimal if for every other point (x_k, y_k, z_k) in this set, point has a smaller coordinate $x_i < x_k$, or $y_k < y_i$, or $z_i < z_k$.
Can a number of minimal points be infinite?

Solution. This is done by induction over dimension.
For 1-dimensional case it is trivial – in each subset of natural numbers there is one minimal element.
For 2-dimensional case choose one minimal point (x_1, y_1). For each x between 1 and x_1 there can be no more than 1 minimal point with that x. For each y between 1 and y_1 there can be no more than 1 minimal point with that y. There can be no minimal points such that $x > x_1$ and $y > y_1$, so we have no more than finite number $(x_1 + y_1)$ of minimal points in plane.

Now for 3 dimensional case. Choose one minimal point (x_1, y_1, z_1).
For each x between 1 and x_1 there is only finite number of minimal points with that x. For each y between 1 and y_1 there is only finite number of minimal points with that y. For each z between 1 and z_1 there is only finite number of minimal points with that z.
There can be no minimal points such that $x > x_1$ and $y > y_1$ and $z > z_1$, so we have no more than finite number of minimal points. QED.

Remark. Of course this induction can continue to higher dimensions.

3. Show that there is a point in the plane, such that distances from it to all integer points are different. (A point (*x,y*) is called integer if *x* and *y* are integer).

First solution. Take $\left(\sqrt{2},\sqrt{3}\right)$, and show it works. Suppose two points $\left(x_1,y_1\right)$ and $\left(x_2,y_2\right)$ give the same distance, then

$$\left(x_1-\sqrt{2}\right)^2+\left(y_1-\sqrt{3}\right)^2=\left(x_2-\sqrt{2}\right)^2+\left(y_2-\sqrt{3}\right)^2$$

$$x_1^2+y_1^2-2x_1\sqrt{2}-2y_1\sqrt{3}=x_2^2+y_2^2-2x_2\sqrt{2}-2y_2\sqrt{3}$$

$$\left(x_1^2+y_1^2-x_2^2-y_2^2\right)+\left(2x_2-2x_1\right)\sqrt{2}+\left(2y_2-2y_1\right)\sqrt{3}=0$$

Lemma 1. If $a+b\sqrt{2}+c\sqrt{3}=0$ for some rational numbers *a, b, c* then *a*=*b*=*c*=0.

Lemma 2. $\sqrt{2},\sqrt{3},\sqrt{\frac{3}{2}}$ are irrational numbers.

From lemma 1 it follows that in our case $x_2=x_1, y_2=y_1$ so the points are actually the same. So it remains to prove lemma 1, but first we shall prove lemma 2.

Proof of lemma 2. If $\sqrt{2}=\frac{m}{n}$ then square it and multiply by denominator: $n^2=2m^2$, then in the left hand side you get even power of 2, and on the right hand side the odd power of 2 (in prime decomposition), it contradicts unique decomposition theorem.

The same will happen with $\sqrt{\frac{3}{2}}$ and $\sqrt{3}$, but in the last case you should count powers of 3 (odd on one side, even on the other).

Proof of lemma 1. $a+b\sqrt{2}+c\sqrt{3}=0$. If all 3 numbers *a, b, c* are 0, that's it, if 1 of them is nonzero its nonsense, if two of them are nonzero, we contradict lemma 2 which was proved already. The only remaining case, that we have yet to exclude, is when all 3 are nonzero.

$a+b\sqrt{2}=-c\sqrt{3}$, square both sides, you get $a^2+2b^2+2ab\sqrt{2}=3c^2$.

But $2ab\neq 0$, hence $\sqrt{2}=\frac{3c^2-a^2-2b^2}{2ab}$ is a rational number, contradiction.

Second solution. A very natural solution to a more general question belongs to a French mathematician Baire (1874-1932).

A set is called ***dense*** if any ball (say, in $\mathbf{R}^n$) contains its point.
A set is called ***nowhere dense*** if any ball contains a smaller ball which is disjoint to given set.
A set is called ***meager*** or of ***first (Baire's) category*** if it is a countable union of nowhere dense sets.
A set is called of ***second (Baire's) category*** if it is not of the first category.

Examples:
A set of rational points is dense, though it is countable.
A sphere (set of points on given distance from the given point), or a line in the plane or a hyperplane in $\mathbf{R}^n$ are nowhere dense.
Any countable set is meager. A set of points, on rational distance from at least one rational point, is meager. A set of perpendicular bisectors to all intervals with rational ends is meager.
A union of 2 meager sets is meager.

A ball, a full cube, and a complement to any meager set are of second category. Why? Because of

Baire's theorem. A complement of meager set is dense.

So, we can choose a point which is on different distance from all rational points, since union of (countable number of) all perpendicular bisectors to intervals with rational ends is of first category. You can also demand that all those distances would be irrational, transcendental, and add any other countable number of other "nowhere dense demands".

Proof of Baire's theorem. A meager set M is a union of countable number of nowhere dense sets A_1, A_2, A_3, ...Take any ball B. It has a sub-ball B_1 disjoint from A_1. B_1 has a sub-ball B_2 disjoint from A_2. B_2 has a sub-ball B_3 disjoint from A_3, and so forth. The intersection of all those balls has a point which is not in M. Since we found it in arbitrary ball B, complement of M is dense. QED.

4. (a) Is it possible to find 1000000 points in the plane, not all of them on one line, so that the distance between each two is integer?
(b) Is it possible to choose an infinite set of points in the plane, not all of them on one line, so that the distance between each 2 is integer?

(a) First solution.
Yes. There are infinitely many Pythagorean triples $a^2 + b^2 = c^2$, such as $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$.
So the distance between points $B(0,1)$ and $A_k = ((k^2 - 1)/2k, 0)$ is rational, for any k between 1 and 1000000.
If we multiply all things by common denominator, or 1000000!, all coordinates and distances will now be integer. Those are 1000000 points, and not all of them are on one line.

Second solution. (Agnis Andjans) Construct a small angle α with rational sine and cosine. To do this, take an angle in Pythagorean triangle with $m \sim n$.
Now take points on unit circle at angles $0, 2\alpha, 4\alpha, 6\alpha, 8\alpha, \ldots$

The distances between those points are $2\sin(k\alpha)$, and those are rational, since it is easy to prove by induction over k, that $\sin(k\alpha)$ and $\cos(k\alpha)$ are polynomials with integer coefficients in $\sin(\alpha)$ and $\cos(\alpha)$ hence they are rational numbers. By the way, coordinates of those points are also rational and for the same reason. If we multiply by common denominator, we get integer points, and all distances are rational, and no 3 points are on one line, since all are on one circle.

(b) No. Consider 3 non-collinear chosen points A, B, C. Every other chosen point D is on integer distance from A and B, so $|AD - BD| = k$ where k is a natural number not exceeding AB, by triangle inequality.
A locus of points D which satisfy $|AD - BD| = k$ is a hyperbola (or a line), so D belongs to a finite family of (less than AB) hyperbolas (with foci A and B), or to line AB, or to perpendicular bisector of AB.
For the same reason D belongs to a finite family of different hyperbolas whose foci are A and C, or to line AC, or to perpendicular bisector of AC.
Intersection of two different hyperbolas have no more than 4 common points (see targil 1), two different lines no more than one, and a straight line intersects a hyperbola in 2 points at most, so we have only a finite number of points.

5*. (a) We have a family of subsets of a countable set (say, natural numbers).
Each two members of the family have no more than 100 common elements. Prove that the family is of countable size (at most).
(b) We have a family of subsets of a countable set, such that intersection of each 2 is finite. Can this family have more than a countable number of elements?

Solution. (a) Suppose not. Take an uncountable family of subsets.
Firstly, we can assume that all of them are infinite, because there are only $\aleph_0$ finite subsets in a countable sets, so if there are finite sets, it won't hurt to exclude them.
Consider all pairs (subset, element) so that subset contains element. Since there are only $\aleph_0$ elements, and more than $\aleph_0$ subsets, one element should belong to more than $\aleph_0$ subsets. Denote this element a_1, and consider subsets that contain it. For the same reasons, more than $\aleph_0$ subsets should have yet another common element, a_2. For the same reason, more than $\aleph_0$ subsets containing a_1, a_2 should have another common element, a_3, and so on.
After you keep on this induction long enough you conclude that more than $\aleph_0$ subsets contain 1000001 common points.

(b) First solution. Yes. Consider a countable set **P** of all points in plane with positive integer coordinate. For each positive real number a define a subset of this set of all points that satisfy $[ax] = y$.
It is easy to see that the intersection of two sets defined by two different numbers is finite, and there are continuum subsets.

Second solution. For any real number, choose a sequence of rational numbers which converges to it. It is a subset of countable set of rational numbers, and each two have final intersection.

Remark. It seemed to me at first that those two solutions are completely different, but they are actually the same. Rational numbers can be visualized as points of integer lattice, y is nominator and x is denominator, and a sequences of points we constructed gives a sequence of ratios converging to the slope.