Targil 4. This time – combinatorics.

1. Six singers came to a festival. Each day several singers make a performance at the scene, while others listen from the audience.
How to organize the shortest possible festival, so that each singer will listen to every other singer?

Solution. Well, the question for 6 is too simple, the answer is 4, so of course we want to formulate the solution in a generalizable way.
We shall apply "duality". Speaking vaguely, we shall not see the days of the festival as subsets singers, but vice versa, the singers as subsets of days. Strictly speaking, each singer defines a subset of the days of the festival, those are days when he sings. A forbidden situation is when singer A was singing on all days when singer B was singing, that is, one subset is contained in another.
So, the question can be reformulated as follows: how big should be the set, so that it has a family of 6 subsets, none of which contains another?
One way to form such a family of subsets is to take subsets of given size. The number of such subsets is a binomial coefficient. The greatest number in each line of Pascal's triangle is the middle one (or the closest to the middle). For example, a set of 4 has a family of 6 subsets of 2 elements each.

So, to prove the general claim (for any number of singers/days) we must find the number of maximal family on mutually not-containing subsets in a set of days. We shall prove that the example we gave is a maximal.
Assume we have a family of mutually not-containing subsets in the set of size N, and they are of different size. Of course, this family can be replaced by another family, of the compliments to the original family, and it will have the same properties. So we may assume that some subsets have more than N/2 elements. Suppose the largest subsets in the family have M elements.
Let us replace all the subsets of M elements by all the subsets of M – 1 elements, which are subsets of those subsets of M elements. The new subsets will be mutually non-containing between themselves and with the other members of the families. The question is, how this replacement affected the number of the subsets?
Each set of order M contains precisely M subsets of M – 1 elements. Each subset is counted no more than N – M + 1 times, but $M > N/2$, Hence $N - M + 1 \leq M$. Hence the number of subset was not decreased by such a replacement. Hence, we can assume no subset is bigger than N/2. By symmetric procedure (or taking the compliment and repeating the same procedure) we conclude that all subsets should have the same size, [N/2].

2. Once upon a time, 17 cannibals have gathered together for a feast. During the feast, some cannibals were killed and eaten by some other cannibals.
It is known however, that there were no 5 cannibals such that neither of them has eaten any of the other 4.
Prove that there is a chain of 5 cannibals, such that each (except the 5^{th}) was eaten by the next member of the chain.

Solution. A cannibal will be called ***kind*** if he didn't eat anybody.
The cannibal is of the ***second kind***, if he ate only kind cannibals at the feast.
The cannibal is of the ***third kind***, if he ate cannibals of the second kind, and maybe also some who were kind, but no others.
Similar we define cannibal of ***k^{th} kind***, those who ate some cannibals of ***k-1^{st}*** kind, maybe of lower kind also, but no others.
If there is a cannibal of the 5^{th} kind, then he ate someone of the 4^{th} kind, who ate someone of the 3^{rd} kind, and so on, which is a chain of 5 cannibals who ate each other. If there is no cannibal of the 5^{th} kind, then there are only 4 kinds of cannibals, so there is a kind of cannibals for there are at least 5 cannibals of that kind. Those 5 cannibals didn't eat each other.

Remark. This fact is known in combinatorics as ***Dilworth's lemma.*** Here is a more scientific way to formulate it: a set of AB+1 elements with partial order must have a chain of length A+1 or an anti-chain of length B+1.

Remark. A proof might be alternatively formulated as follows:
A cannibal is considered cool if he survived the feast. He is considered almost cool if he was eaten only by the cool people, and so on…

3. (a) A graph of 10 vertices doesn't contain 3 vertices which are all adjacent to each other. How many edges (at most) can it have?
(b) A graph of 10 vertices doesn't contain 4 vertices which are all adjacent to each other. How many edges (at most) can it have?

Solution

(a) The answer is 25. It is achieved when by a bipartite graph (גרף דו-צדדי), 5 vertices on each side, all the 5 left vertices are connected to all 5 right vertices, but left vertices, as well as right vertices, are not connected among themselves. The hard part is the proof of maximality.

It is also true that the best result for any number of vertices for a graph which doesn't contain triangle is a bipartite graph, whose sides are as equal as possible: for 2N vertices there are N vertices on each side, for 2N+1 vertices there are N vertices on one side and N+1 on the other.

We shall prove that the bipartite graph is the best by induction.
Suppose we have proved it for graphs with fewer vertices than N. We are given a graph **G** with N vertices, many edges and no triangles, we have to prove that the number of edges is not bigger than it would be for a bipartite graph **B**.
The degree of each vertex of **B** is at least [N/2]. If we find a vertex of whose degree not greater than [N/2], we win. Indeed, we can erase two vertices, one from **G** and another from **B**. After that, bipartite graph will have not less edges by the assumption of induction, but it lost not less edges.

So, it is enough to show that for a graph **G** of N vertices with no triangles, there is a vertex of degree no more than N/2. If not, than the degree of each vertex is (N+1)/2 at least. It means that each vertex has at least (N+1)/2 neighbors, and at most (N – 3)/2 non-neighbors. So each two adjacent vertices have at least one common neighbor, hence we get a triangle.

(b) The same can be proven for a general case. In graph theory, a complete subgraph is called ***a clique***. So, the general question is: given a graph of N vertices doesn't contain K-cliques, how many edges can it have, at most? The answer is: split N vertices in K – 1 subsets of almost equal size, and connect all pairs of vertices from different subsets, and that is the maximum.

The proof is by induction, same as (a). Suppose a graph with no N-cliques surpasses our construction. If we have a vertex of degree no more than $N\frac{K-2}{K-1}$, erase it and apply the induction assumption. If all vertices have greater degree, then each vertex has less than $\frac{N}{K-1}-1$ non-neighbors.
Than choose the vertices, one-by-one, so that each vertex is a neighbor of all previously chosen vertices. Even after we have chosen K–1 vertices, all their non-neighbors a fewer than $N-(K-1)$, so we can choose a clique of K.

4. Yoav, when he was at school, once participated in a test in Arabic language (mevhan bearavit). During the test, he should have connected 10 Hebrew words to their 10 Arabic translations. He didn't know Arabic at all, so he has chosen the translation at random. His mark was 0, which means he had no correct guesses.
Many years later he asked me a question, and it is also my question to you. What was a probability of getting 0 at that test?

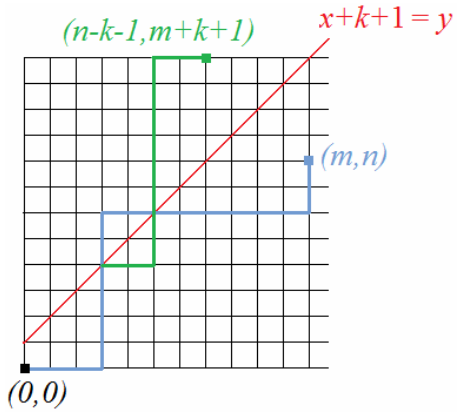
Proof. There are 10! permutation from which Yoav could choose. We shall use inclusion/exclusion principle to count how many permutations will be marked by 0. The answer is number of zero-mark permutations / 10!
Take all 10! permutation and subtract the permutation with one correct guess. There are 10 words that can be guessed correctly, foe each word there are 9! ways to be correct. So 10! – 9!10, now add double intersections, subtract triple intersections, and so on.
There are $\binom{10}{2}$ double intersections, each having 8! elements, $\binom{10}{3}$ triple intersections, each having 7! Elements, and so on, in general $\binom{10}{k}$ each having $(10-k)!$ elements. So we have $10!-9!10+8!\binom{10}{2}-7!\binom{10}{3}+\ldots$
And if we divide it by 10! we get the probability:

$$1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}-\frac{1}{7!}+\frac{1}{8!}-\frac{1}{9!}+\frac{1}{10!}\approx\frac{1}{e}$$

Well, it is a rational number of course, and 1/*e* is irrational (why?), but it is a very good approximation, the best we could get with denominator 10!, which is 8 digits long, so it has precisely the same first 8 digits as 1/*e*: 0.3678794.

5.** There is a long line of $m+n$ people, which all want to buy an ice-cream. The price is 5₪, and out of those people m have a coin of 5₪, and n have a coin of 10₪, and they stand in random order. In the beginning, the cashier has k coins of 5₪. What is the probability that they won't get stuck?

Solution. The solution is, surprisingly, geometrical. There are $\binom{m+n}{m}=\frac{(m+n)!}{m!\,n!}$ ways to arrange the people in the line, we should count the number of all good ways and divide it by the number of all ways.
The key idea is to count the number of bad ways, instead of counting the number of good ways, and then we can subtract.
The ways to arrange people can be seen as ways to have a shortest walk in a city where all streets are parallel or orthogonal to each other, when you have to pass m blocks from west to east and n blocks from south to north, suppose between point (0,0) and a point (m,n).
Getting stuck means arriving to a point (x,y) which is on certain diagonal, on which "the cashier would have spent $k+1$ coins of change", that is $x+k+1=y$.
The bad ways are ways which pass through that "red" diagonal.
Let's reflect the part of the way after its first time it touches the red diagonal with respect to the red diagonal.
We shall have a way which goes from (0,0) to $(n-k-1,m+k+1)$, which is the point symmetric to (m,n) with respect to the red diagonal.
This reflection gives one-to-one correspondence between all bad ways to (m,n) and all ways to $(n-k-1,m+k+1)$, so their number is $\binom{m+n}{m+k+1}$.



Hence the number of good ways is

$$good(k,m,n)=\binom{m+n}{m}-\binom{m+n}{m+k+1}.$$

So the chance is $1-\binom{m+n}{m+k+1}\Big/\binom{m+n}{m}$.

Remark. The special case of the ice cream problem, the number of good ways when $n=m$, $k=0$ has a name: *Catalan numbers*,

$$c_n=\binom{2n}{n}-\binom{2n}{n+1}=\frac{(2n)!}{n!n!}-\frac{(2n)!}{(n-1)!(n+1)!}=(2n)!\frac{n+1-n}{n!(n+1)!}=\frac{(2n)!}{n!(n+1)!}$$

Catalan numbers have a lot of combinatorial descriptions: number of legal sequences of n left and n right brackets, number of minimal triangulations of a convex polygon of n+2 vertices, several description with trees, number of ways to build a tower of base length n from cylindrical sticks, ant others.

6.** A rectangle is called ***good***, if one of its sides is integer. A big rectangle is divided into a finite number of smaller rectangles, which are all good. Show that the big rectangle is also good.

First solution. Firstly, it is obvious that all sides of small rectangles are parallel to the sides of the large rectangles (since if rectangle is parallel, then all its neighbors are parallel).
Consider integral $\iint e^{2\pi ix} e^{2\pi iy} dxdy$ over each rectangle. It is easy to verify that it is 0 iff the rectangle is good. The integral over the big rectangle is a sum of integrals over small rectangles, so it is 0. Hence the big rectangle is good.

Second solution. We shall translate the first solution into a more elementary language. Let $f(x)$ be a function such that $f(x) = f(y)$ iff $x - y$ is integer.
It can be $e^{2\pi ix}$ or, to keep it more elementary, $\{x\}$, the fractional part of x.
Define a function of rectangle

$$F(rectangle) = (f(\max x) - f(\min x)) \times (f(\max y) - f(\min y))$$

F is 0 iff a rectangle is good. F is an additive function, that is, if a rectangle R is divided into two rectangles, R_1 and R_2, then $F(R) = F(R_1) + F(R_2)$
Hence, is small rectangles are good, F each of them is 0, then F of big rectangle is also 0, so the big rectangle is good.
(There is a subtle point here: for some partitions you can not unite any two rectangles into one, but then you can split them in parts and unite them afterwards).

Third solution. Paint the big rectangle in checker-board coloring, starting from the bottom-left corner, such that the sides of checker squares will be ½.
It is easy to see that for each good rectangle its black area is equal to its white area. It is also easy to see that vice versa is true if one corner of rectangle is also a corner of a checker-square (why?).
In the big rectangle black area = white area, since it is divided into parts, each having black area = white area. So the big rectangle is good.

Fourth solution. Lets prolong all the sides of all rectangles to intersect with the sides of big rectangle. Introduce coordinate axis along the sides of big rectangle (left-bottom corner is (0,0)).
Take all lines, which have non-integer coordinates (both vertical and horizontal), and move them by the same distance d and in the same direction. All integer sides of small rectangles will remain the same. But

non-integer sides might change by d, so their area will change by ad. So the change of area, of each small rectangle and total, is a linear function of d. But if it would have all non-integer sides, it would be a quadratic function, that would give a contradiction.

The next solution is in the based on Euler's path method, which was developed by Euler to solve the riddle of Koenigsberg bridges:
http://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg

Fifth solution. Consider a graph: its vertices are vertices of small rectangles, its sides are integer sides of small rectangles. If a rectangle has 4 integer sides, only the horizontal sides will be considered to be edges of the graph. If two rectangles have a common integer side, we should take it twice.
It is easy to see that degrees of all corners of the big rectangle are 1, while the degrees of all other vertices are even.
Start traveling from a left-bottom corner along the edges of the graph, burning the bridges behind you (never repeat the edge which was taken before). At some moment, you will have to stop. It can happen only at a vertex of odd degree, which can only be another corner. But the increment of each coordinate is integer on each step. And one of coordinates of total increment is either width or height.

Sixth solution. From double-counting, we see that each graph has even number of vertices of odd degree. Since sum of all degrees of all vertices is twice the number of edges. Consider the graph that was constructed in the previous solution, and consider in it the connected component of the left-bottom corner. It should contain another vertex of odd degree, so one more corner. But all elements of its connected component have integer coordinates, so either width or height is integer.

Next solution will be based on a topological fact:
Lemma. Suppose a rectangle is painted in 2 colors, black and white. Then you can either find a white path from left to right, or a path from top to bottom.

Proof of lemma. Paint the area to the right of the rectangle white. Consider the white connected component of the white spot near the white side. If it reaches the left side – cool, we have a white path from left to right. Suppose

it doesn't. Then the border of this white spot is all black. And that is the black path from top to bottom.

Seventh solution. Divide all good small rectangles into W-type (those have integer width) and H-type (all the others). Paint the interior of W-type rectangles white and the interior of H-type rectangles black.
Paint the interior part of all horizontal sides of all small rectangles black, and the interior part of all vertical sides of all small rectangles white.
The only unpainted points now are cross-intersections, paint them black, it doesn't matter.
If we have a white path from left to right, it goes along W-type rectangles between vertical sides, so it gives an integer increment of x. So, in this case, width is integer.
If not, by the lemma, we get a black path from top to bottom. In this case, for the same reason, the height is integer. QED.

The last solution is in the spirit of algebraic topology. In algebraic topology, there is a nice notion of homology. It has many definition, one is based on chains. A ***k-chain***, roughly speaking, is a k-dimensional oriented piece of a space, or k-dimensional polytope, or a formal linear combination of such. There is a notion of a ***boundary*** (sum of boundary pieces) which is k-1-chain and of a ***cycle*** – a chain whose boundary is 0.
If there is a mapping from one space to another, the chains can be pushed from the first space to the second.

Eighth solution. Consider each rectangle as a 2-chain. We can consider it on the plane or on the torus, since there is a natural mapping from $\mathbb{R}^2$ to the torus, which is factorization over $\mathbb{Z}^2$.
For rectangular 2-chains and their formal sums, define operator ***horizontal boundary***, which is upper side minus lower side. It is obvious that the rectangle is good if and only if its horizontal boundary is a cycle on the torus. But horizontal boundary is additive, and sum of cycles is a cycle
The horizontal boundary of the big rectangle is sum of horizontal boundaries of the small rectangles, so it is cycle, so the big rectangle is good.