Factorials – solutions.

1. (a) Prove that $\int_0^\infty e^{-x}x^N dx = N!$ for any natural *N*.

(b) Prove that $\int_0^1 x^K (1-x)^M \, dx = \frac{K!M!}{(K+M+1)!}$ for natural *K*, *M*.

Solution. (a) Induction. For *N*=0 You get integral = 0.

The step of induction: proving that $\int_0^\infty e^{-x}x^N dx = N\int_0^\infty e^{-x}x^{N-1}dx$.

Apply integration by parts. First function is x^N, the derivative of the second is e^{-x}. Minus (of deriving e^{-x}) times minus (of integration by parts) is plus.

(b) I think that the first solution (I saw it in Euler's book) was integration by parts (integrate x^K, differentiate $(1-x)^M$).

However, there is another classical solution I like better.

Consider double integral over quadrant (¼ plane):

$$I(K,M) = \int_0^\infty \int_0^\infty x^K y^M e^{-x-y} dxdy$$

We shall compute it in two ways.

First way: split it into product of two integrals.

$$I(K,M) = \int_0^\infty \int_0^\infty x^K y^M e^{-x-y} dxdy = \int_0^\infty x^K e^{-x} dx \int_0^\infty y^M e^{-y} dy = K!M!$$

Second way: substitute $z = x+y, t = \frac{x}{x+y}$ which is $x = zt, y = z - zt$

$$I(K,M) = \int_0^\infty \int_0^\infty \left(\frac{x}{z}\right)^K \left(\frac{y}{z}\right)^M z^{K+M} e^{-x-y} dxdy = \int_0^\infty \int_0^\infty t^K (1-t)^M z^{K+M} e^{-z} dxdy$$

To continue this computation we need to convert *dxdy* into *dzdt* which is done by the Jacobian.

$$J = \begin{vmatrix} \partial x/\partial z & \partial x/\partial t \\ \partial y/\partial z & \partial y/\partial t \end{vmatrix} = \begin{vmatrix} t & z \\ 1-t & -z \end{vmatrix} = \begin{vmatrix} t & z \\ 1 & 0 \end{vmatrix} = -z$$

So $dxdy = |J|dzdt = zdzdt$, hence

$$I(K,M)=\int_0^\infty\int_0^\infty t^K(1-t)^M z^{K+M}e^{-z}dxdy=\int_0^1\int_0^\infty t^K(1-t)^M z^{K+M+1}e^{-z}dzdt=$$

$$=\int_0^\infty z^{K+M+1}e^{-z}dz\int_0^1 t^K(1-t)^M dt=(K+M+1)!\cdot\int_0^1 t^K(1-t)^M dt$$

The two answers for the same questions that we have computed in two ways should be equal, hence $(K+M+1)!\cdot\int_0^1 t^K(1-t)^M dt=K!M!$

QED.

Remark. First part of the question gives a natural extension of $N!$ to all real non-negative numbers, or in fact to all complex numbers for which $\mathrm{Re}N\geq 0$. In fact, it can be extended to all complex plane, and it will have poles (infinite values) at all negative integers. The standard notation is

$$\Gamma(N)=\int_0^\infty e^{-x}x^{N-1}dx=(N-1)!$$

and it is called "gamma function" or "Euler's gamma function".
Different mathematicians have found some nice properties of Γ and proved several theorems of the type "why Γ is the most natural extension of $(N-1)!$ to the positive/complex numbers".
Here are some colored pictures http://mathworld.wolfram.com/GammaFunction.html

As for the second part, it allows to build the natural extension of binomial coefficients to non-integer number. The integral we asked about is usually denoted B(K+1,M+1) and called "beta function".

The formula we proved in standard notation looks so: $\mathrm{B}(p,q)=\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

2. Define (using 1a) and compute ½!

Solution. As it was explained in the above remark, the natural definition is $(1/2)!=\int_0^\infty e^{-x}\sqrt{x}dx$. I don't believe there is an elementary function, whose derivative is $e^{-x}\sqrt{x}$. So, we need a trick.
Substitute $K=M=$ ½ in the formula 1(b):

$$2!\cdot\int_0^1\sqrt{t}\sqrt{1-t}\,dt=\left(\left(\tfrac{1}{2}\right)!\right)^2$$

So. It is enough to compute $\int_0^1\sqrt{t(1-t)}\,dt$. This integral is related to a circle.

Really, let $u=t-\frac{1}{2}$. Then $t(t-1)=\left(\frac{1}{2}-u\right)\left(\frac{1}{2}+u\right)=\frac{1}{4}-u^2$ and

$\int_0^1\sqrt{t(1-t)}\,dt=\int_{-1/2}^{1/2}\sqrt{\frac{1}{4}-u^2}\,dt$.

This is precisely the are between diameter of a circle of radius ½ and its arc, which is half a circle of radius ½ so it is $\pi/8$.

So, we have $2!\cdot\pi/8=\left(\left(\tfrac{1}{2}\right)!\right)^2$ and hence $\left(\tfrac{1}{2}\right)!=\sqrt{2!\cdot\pi/8}=\sqrt{\pi/4}=\sqrt{\pi}/2$

Remark. The calculator in Windows knows it ☺

3. Compute the volume of the N-dimensional ball of radius 1 and the area of its surface.

Solution. Denote S_N the area of its surface, V_N the volume. Firstly,

$$S_N = NV_N$$

This easy fact can be explained in 2 ways.

First explanation. Divide the surface into small (or infinitesimal) countries. Divide the volume into conic parts, whose vertex is the center of the ball, and bases are countries on the surface. Each *N*-dimensional cone can be computed as *Sh/N*. In our case *h*=1, so each volume part is *N* times smaller than corresponding area part. This constant ratio will be preserved after integration.

Second explanation. To compute volume of the ball, we shall split it into thin concentric spherical layers (like cabbages). At radius *R* we have layer of width *dR* and area S_NR^{N-1}, and integration of this gives

$$V_N=\int_0^1 S_N R^{N-1}dR=\frac{S_N}{N}.$$

So, it is sufficient to compute the area or the volume and not both.

First solution. Let z be one of the coordinates. The hyperplane at level z intersects the ball along a thin layer of radius $\sqrt{1-z^2}$, hence its $N-1$-dimensional volume is $V_{N-1}\cdot\left(\sqrt{1-z^2}\right)^{N-1}$. The total volume is given by the integral of this expression times dz.

$$V_N=\int_{-1}^{1}V_{N-1}\cdot\left(\sqrt{1-z^2}\right)^{N-1}dz$$

This integral begs for trigonometric substitution:

$$z=\cos x$$
$$dz=-\sin x dx$$
$$\sqrt{1-z^2}=\sin x$$

Then we get

$$V_N=V_{N-1}\int_{-\pi}^{\pi}\left(\sin x\right)^N dx$$

So now we have to solve 4(a) and multiply the answers.
The details are left to the reader (if he or she will really want to finish the first solution after they see the second one).

Second solution. Consider N-dimensional integral:

$$G_N=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}e^{-\left(x_1^2+x_2^2+\ldots+x_n^2\right)}dx_1dx_2\ldots dx_N$$

We shall compute it in 2 ways. First way is splitting it into product:

$$G_N=\int_{-\infty}^{\infty}e^{-x_1^2}dx_1\cdot\int_{-\infty}^{\infty}e^{-x_2^2}dx_2\cdot\ldots\cdot\int_{-\infty}^{\infty}e^{-x_n^2}dx_N=G_1^N$$

Second way is integrating it thing cabbage-like spherical slices of radius R centered at zero. The volume of each slice is $S_nR^{N-1}dR$ so we get

$G_N=\int_0^{\infty}e^{-R^2}S_nR^{N-1}dR$, now substitute $u=R^2, du=2RdR$:

$$G_N=\frac{S_n}{2}\int_0^{\infty}e^{-R^2}R^{N-2}2RdR=\frac{S_n}{2}\int_0^{\infty}e^{-u}u^{\frac{N}{2}-1}du=\frac{S_n}{2}\Gamma\left(N/2\right)=\left(\frac{N}{2}-1\right)!\frac{S_n}{2}$$

So, $G_1^N=\left(\frac{N}{2}-1\right)!\frac{S_n}{2}$ and $G_1^N=\left(\frac{N}{2}\right)!\cdot V_n$. To conclude this solution, we have yet to find G_1.

Instead of computing it directly, use the fact that we know the area of the unit circle in plane. $G_1^2=\pi$. **The answer:** $\boxed{V_N=\frac{\pi^{N/2}}{(N/2)!}\quad,\quad S_N=N\frac{\pi^{N/2}}{(N/2)!}}$.

Remark. And if someone asks what is $(1/2)!$ or $(3/2)!$ we say that

$(-1/2)!=\sqrt{\pi}$ and $(N+1)!=N!(N+1)$

hence $(1/2)!=\frac{\sqrt{\pi}}{2}$, $(3/2)!=\sqrt{\pi}\cdot\frac{1}{2}\cdot\frac{3}{2}$, $(5/2)!=\sqrt{\pi}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}$.

Of course, for $N=2k+1$ we can write the formula in usual notations

$$V_N=2\frac{(2\pi)^{(N-1)/2}}{1\cdot 3\cdot\ldots\cdot N}\quad,\quad S_N=2\frac{(2\pi)^{(N-1)/2}}{1\cdot 3\cdot\ldots\cdot(N-2)}\ ,$$

where the denominators contain products of first odd numbers.

4*. (a) Let $I_N=\int_0^{\pi}\sin^N(x)\,dx$. Compute I_N for each integer N.

(b) Show that $\sqrt{N\pi}<4^N\Big/\binom{2N}{N}<\sqrt{\left(N+\frac{1}{2}\right)\pi}$

Solution. (a) Use integration by parts. $\sin^N(x)$ is a product of 2 functions: first , $\sin^{N-1}(x)$, second $\sin(x)$. We will differentiate the first and integrate the second. The integral of sin is - cos, but minus will cancel out with minus of integration by parts. The value at the limits is 0, hence:

$$I_N=\int_0^{\pi}\sin^{N-1}(x)\sin(x)\,dx=(N-1)\int_0^{\pi}\sin^{N-2}(x)\cos(x)\cos(x)\,dx=$$

$$=(N-1)\int_0^{\pi}\sin^{N-2}(x)\left(1-\sin^2(x)\right)dx=(N-1)\left(I_{N-2}-I_N\right)$$

Hence $I_N=(N-1)\left(I_{N-2}-I_N\right)$ so $NI_N=(N-1)I_{N-2}$ or $I_N=\frac{N-1}{N}I_{N-2}$.

It is easily computed that $I_0=\pi$ and $I_1=2$.

So, $I_{2K}=\frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdot\ldots\cdot\frac{1}{2}\cdot\pi$, and $I_{2K+1}=\frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\cdot\ldots\cdot\frac{2}{3}\cdot 2$.

Remark. To write those formulas shortly, special notations were invented. The product of all natural numbers up to *N* having the same parity as *N* is denoted ***N*!!** . In this notation, $I_{2K} = \frac{(2k-1)!!}{(2k)!!} \cdot \pi$ and $I_{2K+1} = \frac{(2k)!! \cdot 2}{(2k+1)!!}$.

(b) The value of sin is not bigger (and usually smaller) than 1. Hence the value of $\sin^N(x)$ decreases as *N* grows, at least on the interval $[0,\pi]$. That is why I_N is decreasing sequence. So, we can write

$$I_{2N} > I_{2N+1} > I_{2N+2}.$$

One obvious consequence of this is so-called Wallis formula.
Since $I_{2N} \approx I_{2N+2}$, and is I_{2N+1} between, $I_{2N} \approx I_{2N+1}$ hence

$$\frac{\pi}{2} \approx \frac{((2N)!!)^2}{(2N-1)!!(2N+1)!!} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\ldots$$

But we shall pay more attention and obtain more precise inequalities.
First we shall reduce !! to ! .

$$(2N)!! = 2\cdot 4\cdot 6\cdot\ldots\cdot 2N = 2^N\cdot 1\cdot 2\cdot 3\cdot\ldots\cdot N = 2^N\cdot N!$$

$$(2N+1)!! = \frac{(2N+1)!}{(2N)!!} = \frac{(2N+1)!}{2^N\cdot N!}$$

Now lets develop $I_{2N} > I_{2N+1} > I_{2N+2}$:

$$\frac{(2N-1)!!}{(2N)!!}\cdot 2\pi > \frac{(2N)!!\cdot 2}{(2N+1)!!} > \frac{(2N+1)!!}{(2N+2)!!}\cdot 2\pi$$

$$(2N+1)\cdot\frac{\pi}{2} > \frac{((2N)!!)^2}{((2N-1)!!)^2} > \frac{(2N+1)^2}{2N+2}\cdot\frac{\pi}{2}$$

$$\left(N+\frac{1}{2}\right)\cdot\pi > \frac{((2N)!!)^4}{((2N)!)^2} > \frac{4N^2+4N}{2N+2}\cdot\frac{\pi}{2}$$

$$\left(N+\frac{1}{2}\right)\cdot\pi > \left(\frac{(N!\cdot 2^N)^2}{(2N)!}\right)^2 > N\cdot\pi$$

$$\sqrt{\left(N+\frac{1}{2}\right)\cdot\pi} > \frac{4^N}{\frac{(2N)!}{(N!)^2}} > \sqrt{N\cdot\pi}$$

5. Compute $\lim_{N\to\infty}\left(\frac{\sqrt[N]{N!}}{N}\right)$.

First solution.

Lemma 1. If sequence $\{a_n\}$ has a limit, then the sequence of its averages $\left\{\frac{a_1+a_2+\ldots+a_n}{n}\right\}$ has the same limit.

Proof is direct, with epsilons and deltas. All element of the second sequence after some index are near the limit, and others give finite (and reducing) contribution.

Lemma 2. To find a limit of a sequence $\{b_n\}$ it is enough to find the limit of the sequence $\{(n+1)b_{n+1} - nb_n\}$, and if we do, the limit of $\{b_n\}$ is the same.

Proof : it follows directly from lemma 1, since sequence $\{b_n\}$ are averages of sequence $\{(n+1)b_{n+1} - nb_n\}$.

Lemma 1'. If sequence of positive numbers $\{a_n\}$ has a limit, then the sequence of its geometric means $\left\{\sqrt[n]{a_1a_2\cdot\ldots\cdot a_n}\right\}$ has the same limit.

Lemma 2'. To find a limit of a sequence of positive numbers $\{b_n\}$ it is enough to find the limit of the sequence $\left\{\frac{b_{n+1}^{n+1}}{b_n^n}\right\}$ has the same limit.

The last two lemmas follow from the former two by taking exp of all sequences. We shall use lemma 2', but I wanted to show you other things which are useful.

So, for limit of $\frac{\sqrt[N]{N!}}{N}$ it would be enough to find the limit of

$$\frac{(N+1)!}{(N+1)^{N+1}}\Big/\frac{N!}{N^N}=\frac{N^N}{(N+1)^N}=\frac{1}{\left(1+\frac{1}{N}\right)^N}$$

The denominator is a famous expression for $\boldsymbol{e}$, so this sequence and the original one two converge to $1/e$.

Second solution (I learned it recently from a 13-year old who invented it himself, Gil Gitik). Take the log. $\ln\left(\frac{\sqrt[N]{N!}}{N}\right)=\frac{\ln\frac{1}{N}+\ln\frac{2}{N}+...+\ln\frac{N}{N}}{N}$

This is Riemann sum for an integral $\int_0^1 \ln x dx = \left(x\ln x - x\right)_0^1 = -1$.

So, before taking ln, the limit was e^{-1}.