During the discussion of 3rd targil (factorials) many unexpected ideas were mentioned, mostly by Alexey (Gladkikh), Gal and Oded, several nice new solutions and remarks, so I have decided to write an appendix to the solutions.

1(b). Third solution (Alexey). Choose at random (with uniform probability distribution) a point A, and independently, $M+K$ more points. What is the probability that the point A will hold place $M+1$, meaning, there will be M points before and K points after?
We shall compute it in two ways. First way: well, there are $\binom{M+K}{K} = \frac{(M+K)!}{M!K!}$ ways to choose divide $M+K$ other points into M specific points and K specific points, and now multiply this number if ways by the probability that M specific points will be before and K specific points will be after. If point A is at place x then the probability is $x^M (1-x)^K$, so in general we get integral over probability measure of this dx. In the end we get

$$\frac{(M+K)!}{M!K!}\int_0^1 x^M (1-x)^K \, dx$$

The second way to compute the same thing: all animals are equal, so the probability for point A to be on each place, from 1 to $M+K+1$ is the same:

$$\frac{1}{M+K+1}$$

Hence

$$\frac{(M+K)!}{M!K!}\int_0^1 x^M (1-x)^K \, dx = \frac{1}{M+K+1}$$

So:

$$\int_0^1 x^M (1-x)^K \, dx = \frac{M!K!}{(M+K+1)!}$$

Of course, this solution, as well as integration by parts, works for integer values only.

2. Second solution (Gal & Dan) We shall compute $\left(-\frac{1}{2}\right)!$, and $\left(\frac{1}{2}\right)!=\left(-\frac{1}{2}\right)!\cdot\frac{1}{2}$. Let $u=\sqrt{x}$, then $du=\frac{dx}{2\sqrt{x}}$, hence $2udu=dx$, so:

$$\left(-\frac{1}{2}\right)!=\int_0^\infty\frac{e^{-x}}{\sqrt{x}}dx=\int_0^\infty\frac{e^{-u^2}}{u}2udu=2\int_0^\infty e^{-u^2}du=\int_{-\infty}^{\infty}e^{-u^2}du$$

This integral is classical and equal to $\sqrt{\pi}$. So the answer is

$$\left(\frac{1}{2}\right)!=\left(-\frac{1}{2}\right)!\cdot\frac{1}{2}=\frac{\sqrt{\pi}}{2}$$

Remark. How to compute $\int_{-\infty}^{\infty}e^{-u^2}du$ was explained in the second solution for problem 3. This function, e^{-u^2}, is called ***Gaussian***, ***bell-shaped (פעמון)***, or ***normal distribution*** function, and it plays important role in probability theory, partly because of central limit theorem.
Though we know the total integral under it, we can't get its indefinite integral as an elementary function (it is not arithmetical combination of powers, exponents, logs and trigo), hence a special notation was invented, it (up to some coefficients) is called **erf**(*x*).

3. Third solution (Oded). Like in the first solution, for the volume of the ball we wrote, $V_N=\int_{-1}^{1}V_{N-1}\cdot\left(\sqrt{1-z^2}\right)^{N-1}dz$. So all we need is

$$\int_{-1}^{1}\left(\sqrt{1-z^2}\right)^{N-1}dz=\int_{-1}^{1}\left(\sqrt{1-z}\right)^{N-1}\left(\sqrt{1+z}\right)^{N-1}dz$$

Use a linear substitution which brings interval [-1,1] to interval [0,1]
$u = (z+1)/2$, $du = dz/2$, we get

$$\int_{-1}^{1}\left(\sqrt{1-z^2}\right)^{N-1}dz=\int_{-1}^{1}\left(\sqrt{1-z}\right)^{N-1}\left(\sqrt{1+z}\right)^{N-1}dz=\int_0^1(2u)^{\frac{N-1}{2}}\left(2(1-u)\right)^{\frac{N-1}{2}}2du=$$

$$=2^N\int_0^1u^{\frac{N-1}{2}}(1-u)^{\frac{N-1}{2}}du=\frac{N!2^N}{\left(\left(\frac{N-1}{2}\right)!\right)^2}$$

4. (a) $I_N = \int_0^{\pi} \sin^N(x)\,dx$

Second solution (Alexey)**.** $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$, when we raise it to the power N we get sum of expressions of the kind $\binom{N}{K} e^{(N-K)ix} e^{-Kix}$ all divided by $(2i)^N$.

So, we have to compute $\int_0^{\pi} e^{(N-2K)ix}\,dx$.

If $K=2N$ then we get integral of 1, which is π. In other cases we get

$$\int_0^{\pi} e^{(N-2K)ix}\,dx = \frac{e^{(N-2K)ix}}{(N-2K)i}\Bigg|_0^{\pi}$$

this is 0 for even $N-2K$, that is, for even *N*, and for odd *N* we get.

$$\int_0^{\pi} e^{(N-2K)ix}\,dx = \frac{e^{(N-2K)ix}}{(N-2K)i}\Bigg|_0^{\pi} = \frac{-2}{(N-2K)i}$$

Hence the answer is, for even N:

$$I_N = \int_0^{\pi} \sin^N(x)\,dx = \frac{\sum_{K=0}^{N} \int_0^{\pi} \binom{N}{K} (-1)^K e^{(N-K)ix} e^{-Kix}\,dx}{(2i)^N} = \frac{\binom{N}{N/2}\pi}{2^N}$$

For odd *N*

$$I_N = \int_0^{\pi} \sin^N(x)\,dx = \frac{\sum_{K=0}^{N} \int_0^{\pi} \binom{N}{K} (-1)^K e^{(N-2K)ix}\,dx}{(2i)^N} = \frac{\sum_{K=0}^{N} \binom{N}{K} (-1)^K \int_0^{\pi} e^{(N-2K)ix}\,dx}{(2i)^N} =$$

$$= \sum_{K=0}^{N} \frac{-2(-1)^K \binom{N}{K}}{(2i)^N (N-2K)i} = \sum_{K=0}^{N} \frac{-2(-1)^{\frac{N+1}{2}-K} \binom{N}{K}}{2^N (N-2K)} = \sum_{K=0}^{N} \frac{(-1)^{\frac{N-1}{2}-K} \binom{N}{K}}{2^{N-1}(N-2K)}$$

Second solution (Alexey). Let $t=\sin^2 x$, then $1-t=\cos^2 x$, $\dfrac{dt}{\sqrt{t(1-t)}}=2dx$

$$I_N=\int_0^{\pi}\sin^N(x)\,dx=2\int_0^{\pi/2}\sin^N(x)\,dx=\int_0^1 t^{N/2}\frac{dt}{\sqrt{t(1-t)}}=\frac{\left(\frac{N}{2}\right)!}{\left(\frac{N}{2}-1\right)!\sqrt{\pi}}$$

Because of 1(b).

4(b). $\sqrt{N\pi}<4^N\Big/\binom{2N}{N}<\sqrt{\left(N+\frac{1}{2}\right)\pi}$

An elementary aproach, which gives a less good but similar inequality based on a classical riddle, I recalled it two days after the meeting.

$$4^N\Big/\binom{2N}{N}=\frac{2^{2N}(N!)^2}{(2N)!}=\frac{\left((2N)!!\right)^2}{(2N)!}=\frac{(2N)!!}{(2N-1)!!}$$

Denote $A_N=\dfrac{(2N)!!}{(2N-1)!!}=\dfrac{2}{1}\cdot\dfrac{4}{3}\cdot\ldots\cdot\dfrac{2N}{2N-1}$,

$B_N=\dfrac{(2N+1)!!}{(2N)!!}=\dfrac{3}{2}\cdot\dfrac{5}{4}\cdot\ldots\cdot\dfrac{2N+1}{2N}$.

Since $\dfrac{2}{1}>\dfrac{3}{2}>\dfrac{4}{3}>\dfrac{5}{4}>\dfrac{6}{5}>\ldots>1$ we get, when we multiply such inequalities $2B_{N-1}>A_N>B_N$.

On the other hand, $B_{N-1}A_N=2N$, $B_NA_N=2N+1$ hence $4N>A_N^2>2N+1$.

A more precise inequality was found by Alexey.

$$\boxed{\sqrt{\left(N+\frac{1}{4}\right)\pi}<4^N\Big/\binom{2N}{N}<\sqrt{N\pi+1}}$$

Start by taking the ln:

$$\frac{\ln\pi+\ln\left(N+\frac{1}{4}\right)}{2}<N\ln 4+2\ln(N!)-\ln\left((2N)!\right)<\frac{\ln\pi+\ln\left(N+\frac{1}{\pi}\right)}{2}$$

Let us denote the left L_N, the middle M_N, the right R_N. Lets compute

$$L_{N+1}-L_N \qquad M_{N+1}-M_N \qquad R_{N+1}-R_N$$

We get

$$\frac{\ln\left(N+1+\frac{1}{4}\right)-\ln\left(N+\frac{1}{4}\right)}{2},$$

$$\ln 4+2\ln(N+1)-\ln(2N+1)-\ln(2N+2),$$

$$\frac{\ln\left(N+1+\frac{1}{\pi}\right)-\ln\left(N+\frac{1}{\pi}\right)}{2}$$

Simplify it and multiply by 2

$$\ln\left(1+\frac{1}{N+1/4}\right),\quad 2\left(\ln\left(1+\frac{1}{2N+1}\right)\right),\quad \ln\left(1+\frac{1}{N+1/\pi}\right)$$

If we would prove $L_{N+1}-L_N < M_{N+1}-M_N < R_{N+1}-R_N$ we would get the statement by trivial induction. Unfortunately, the inequalities are vice versa.

Let's prove $\ln\left(1+\frac{1}{N+1/4}\right)>2\left(\ln\left(1+\frac{1}{2N+1}\right)\right)>\ln\left(1+\frac{1}{N+1/\pi}\right)$

$$1+\frac{1}{N+1/4}>\left(1+\frac{1}{2N+1}\right)^2>1+\frac{1}{N+1/\pi}$$

$$\frac{1}{N+1/4}>\frac{2}{2N+1}+\frac{1}{(2N+1)^2}>\frac{1}{N+1/\pi}$$

$$\frac{1}{N+1/4}-\frac{1}{N+1/2}>\frac{1}{(2N+1)^2}>\frac{1}{N+1/\pi}-\frac{1}{N+1/2}$$

$$\frac{1/4}{(N+1/4)(N+1/2)}>\frac{1}{(2N+1)^2}>\frac{\frac{1}{2}-\frac{1}{\pi}}{(N+1/\pi)(N+1/2)}$$

$$\frac{1}{2N+1/2}>\frac{1}{2N+1}>\frac{2-\frac{4}{\pi}}{2N+2/\pi}$$

This is easy.

So, we can't apply standard induction from 1 to infinity, we can only hope to do it vice versa – from infinity to 1. By the easy version of inequality,

$$\lim_{N\to\infty}\sqrt{N}\binom{2N}{N}\Big/4^N=1$$

Hence L_H-M_H, M_H-R_H for huge H are very close to 0, closer than any given positive epsilon, and we have proved that

$(L_{N+1}-L_N)-(M_{N+1}-M_N),(M_{N+1}-M_N)-(R_{N+1}-R_N)$ is always negative, hence

$$L_{K+1}-M_{K+1}=L_H-M_H+\sum_{N=K+1}^{H-1}\left((L_{N+1}-L_N)-(M_{N+1}-M_N)\right)<\varepsilon,$$

$$M_{K+1}-R_{K+1}=M_H-R_H+\sum_{N=K+1}^{H-1}\left((M_{N+1}-M_N)-(R_{N+1}-R_N)\right)<\varepsilon$$

for each positive ε hence $L_{K+1}-M_{K+1},M_{K+1}-R_{K+1}\le 0$ and so $L_K-M_K,M_K-R_K<0$,
QED.

Especially for those who think that was not cool enough, Alexey had improved this inequality even further three days later:

$$\boxed{\sqrt{\left(N+\frac{1}{4}\right)\pi}<4^N\Big/\binom{2N}{N}<\sqrt{\left(N+\frac{1}{4}+\frac{1}{4N+3}\right)\pi}}$$

The idea is the following. Suppose we want to prove a good upper bound for our expression. In the spirit of what was done before, we have to choose a smaller number *t*, such that an inequality

$2\left(\ln\left(1+\frac{1}{2M+1}\right)\right)>\ln\left(1+\frac{1}{M+t+1/4}\right)$ will hold for all $M\ge N$.

He found out that when you choose any t, the expression

$$D_M=2\left(\ln\left(1+\frac{1}{2M+1}\right)\right)-\ln\left(1+\frac{1}{M+t+1/4}\right)$$

is negative for small *M* and positive after it passes certain point.
This point is *N* for which the original inequality can be proven.
To find this point he sets the derivative D_M as a function of *M* to be 0, and gets an expression for t.

5. $\lim_{N\to\infty}\left(\frac{\sqrt[N]{N!}}{N}\right)=\frac{1}{e}$

Gal asked, whether it is possible to prove it using Stirling's formula. I didn't accept it, since this equality is a first step in proving or guessing Stirling's formula.
For the sake of those who couldn't come to the meeting, here we shall give the proof of Stirling's formula.

Stirling's formula is about approximating $N!$
The problem 5 gives a (false) hope, that

$\frac{N!}{N^N} \approx \frac{1}{e^N}$ that is, $\lim_{N\to\infty}\left(\frac{e}{N}\right)^N N! = 1.$

Let us check how far from truth is that.

Denote $A_N = \ln\left(\left(\frac{e}{N}\right)^N N!\right) = N - N\ln N + \ln(N!)$, let

$$B_N = 1 + N\ln N - (N+1)\ln(N+1) + \ln(N+1) = 1 - N\ln\left(\frac{N+1}{N}\right) = 1 - N\ln\left(1+\frac{1}{N}\right)$$

We want A_N to have a limit, it is equivalent to convergence of the series $\sum_{N=1}^{\infty} B_N$. From the expression $B_N = 1 - N\ln\left(1+\frac{1}{N}\right) \approx 0$ we have a hope, but if we look closer

$$B_N = 1 - N\ln\left(1+\frac{1}{N}\right) = 1 - N\left(\frac{1}{N} - \frac{1}{2N^2} + O\left(N^{-3}\right)\right) = \frac{1}{2N} + O\left(N^{-2}\right)$$

So $\sum_{N=1}^{\infty} B_N$ diverges even though B_N goes to 0.

However, would we have

$$B_N = 1 - \left(N+\frac{1}{2}\right)\ln\left(1+\frac{1}{N}\right) = 1 - \left(N+\frac{1}{2}\right)\left(\frac{1}{N} - \frac{1}{2N^2} + O\left(N^{-3}\right)\right) = O\left(N^{-2}\right)$$

Then $\sum_{N=1}^{\infty} B_N$ would converge and A_N would have a limit.

This would happen if we'd have

$$B_N = 1 + \left(N+\frac{1}{2}\right)\ln N - \left(N+1+\frac{1}{2}\right)\ln(N+1) + \ln(N+1)$$

and $A_N = \ln\left(\left(\frac{e}{N}\right)^N \frac{N!}{\sqrt{N}}\right) = N - \left(N+\frac{1}{2}\right)\ln N + \ln(N!).$

So, the expression $\left(\frac{e}{N}\right)^N \frac{N!}{\sqrt{N}}$ has a finite positive limit α, in other words

$$N! \approx \alpha\sqrt{N}\left(\frac{N}{e}\right)^N$$

To find α take Wallis formula, of exercise 4(b) which is a more precise version of Wallis formula.

$$\sqrt{N\pi} \approx \frac{4^N \left((N)!\right)^2}{(2N)!} \approx \frac{4^N \left(\alpha\sqrt{N} \left(\frac{N}{e} \right)^N \right)^2}{\alpha\sqrt{2N} \left(\frac{2N}{e} \right)^N}$$

Surprisingly, all powers of N and e and most powers of 2 cancel out and we get $\alpha = \sqrt{2\pi}$ and Stirling's formula is

$$\boxed{N! \approx \sqrt{2\pi N} \left(\frac{N}{e} \right)^N}$$

It is on of the most beautiful and important formulas about factorials.
The most natural prove, which I showed above, I have read it in a book on probability theory, by a famous French mathematician, Henri Poincare, who lived about a 100 years ago. Stirling's formula has also several other classical proofs.

Gal has remarked, that Wallis formula $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \ldots$ follows from Euler's formula for sine: $\boxed{\frac{\sin(\pi x)}{\pi} = x \cdot \prod_{N=1}^{\infty} \left(1 - \frac{x^2}{N^2} \right)}$.

If you revert Wallis formula you get

$$\frac{2}{\pi} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \ldots = \frac{1 \cdot 3}{2^2} \cdot \frac{3 \cdot 5}{4^2} \cdot \frac{5 \cdot 7}{6^2} \cdot \frac{7 \cdot 9}{8^2} \cdot \ldots = \frac{2^2 - 1}{2^2} \cdot \frac{4^2 - 1}{4^2} \cdot \frac{6^2 - 1}{6^2} \cdot \ldots =$$

$$= \left(1 - \frac{1}{2^2} \right) \cdot \left(1 - \frac{1}{4^2} \right) \cdot \left(1 - \frac{1}{6^2} \right) \cdot \ldots$$

This is precisely what You get when You substitute ½ into Euler's formula.

Another nice conclusion from Euler's formula turns out when You compute the coefficient of x^3, on both sides: $-\frac{\pi^2}{6} = -\sum_{N=1}^{\infty} \frac{1}{N^2}$.

Intuitively, Euler's formula presents $\frac{\sin(\pi x)}{\pi}$ as a polynomial, whose roots are all integer numbers, and hence it is a product over all integer numbers.

Gal has asked for the proof of Euler's formula. Unfortunately, the only proof I know uses complex functions (merukavot) so some of You may not understand it. But anyway, here it comes.

Firstly, let us discuss convergence. In general when you have a product $\prod_{N=1}^{\infty}(1+a_N)$ and the sequence a_N tends to 0. We say that this product ***converges well*** if it converges to a number, which is neither 0 nor infinity.

Lemma. $\prod_{N=1}^{\infty}(1+a_N)$ converges well iff $\sum_{N=1}^{\infty}a_n$ converges.

(iff is short for "if and only if")

Proof of lemma. $\prod_{N=1}^{\infty}(1+a_N)$ converges well iff its log, which is $\sum_{N=1}^{\infty}\log(1+a_n)$ converges. But for small numbers $\log(1+a_n)\approx a_n$. QED.

From this lemma, we see Euler's product converges.

For every function $f(z)$ there is a nice construction, called logarithmic derivative $f'(z)/f(z)$. It is defined well and nicely even when definition of logarithm is messy. A logarithmic derivative of a product is a sum of logarithmic derivatives (this is precisely Leibniz rule).
Logarithmic derivative of Euler's formula (of both sides) gives:

$$\boxed{\pi\operatorname{ctg}(\pi x)=\frac{1}{x}+\sum_{N=1}^{\infty}\left(\frac{1/N}{1+x/N}+\frac{-1/N}{1-x/N}\right)=\frac{1}{x}+\sum_{N=1}^{\infty}\left(\frac{1}{x+N}+\frac{1}{x-N}\right)}$$

It is as beautiful as the original formula itself, and is equivalent to it.
Take derivative of both parts, and change the signs, You will get a third beautiful equivalent formula,

$$\boxed{\frac{\pi^2}{\sin^2(\pi x)}=\sum_{N\in\mathbb{Z}}\frac{1}{(x+N)^2}}$$

It converges even better so the order of summation is not important.

We shall prove the third formula. Both left hand and right hand sides are defined in all complex plane, periodic with period 1, and analytic everywhere except integer points. At integer points they have precisely the

same singularity $\frac{1}{z^2}$ at all integer points, and far from real line both are bounded and tend to 0.

So the difference of the 2 is an analytic bounded function on complex plane, and it is constant by Liouville theorem, 0 in our case since it goes to 0 far from real axis. QED.