This time the problems are about differentiating/integrating in geometrical context.

1. Consider a sphere (the surface of a ball) of radius R, and cut it by a horizontal plane at distance h from its north pole. Compute the area of the part of the sphere which is above the plane (spherical kipa).

Solution: the answer is surprising. It is proportional to h.

For unit sphere, it is $2\pi h$, for sphere of radius R it is $2\pi hR$.

Consider a huge number of densely distributed horizontal planes. They cut the spherical surface into a lot of circular tilted bands.

Consider plane at distance h from the north pole. If the center of the sphere is (0,0,0), this plane can be described analytically as $z = R - h$.

The thin spherical band, between z and $z+dz$, is circular and its length is $2\pi r$ where r I a radius of the circle, $r^2 + z^2 = R^2$.

The height of this spherical band is dz. But its width is bigger, because it is tilted. The quotient between the width and the height is R/r, since triangles are similar.

So the area of a thin spherical band of height dz is length times width = $2\pi r \times dz \times R/r =$

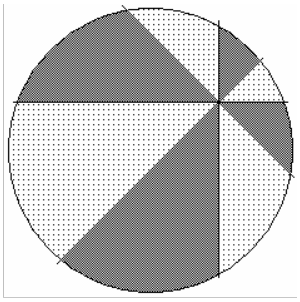
$2\pi R dz$. It is a constant function!

Integrate it over an interval of length h, you get $2\pi Rh$.

2. A circle was cut by 4 lines, passing through the same interior point of the circle and having forming 8 equal angles at that point (of 45 degrees each).

So, the inner part of the circle is divided into 8 parts, which are painted in 2 colors, so that neighboring parts have different colors (like the chessboard).

Show, that each color takes the same part of the area.



First solution. Rotate this set of lines around their common intersection with a constant angular speed v. When one of the lines will be passing through the center of the circle, both colors will occupy the same area (due to symmetry w. r. t. diameter).

So, all we have to prove is that the difference between dark area and light area will be preserved during the rotation.

The area swept by an interval of length l rotated around its end by an infinitesimal angle $d\alpha$ is $l^2 d\alpha/2$.

In our picture, when all 8 the intervals are rotated, 4 of them turn dark area into light, and another for turn light area into dark. Denote the lengths of 8 intervals, a, e, b, f, c, g, d, h clockwise. We compute the derivative of "dark area – light area". During time dt it will sweep angle $d\alpha = v\,dt$ and area $((a^2+b^2+c^2+d^2) - (e^2+f^2+g^2+h^2))d\alpha/2$, so derivative (when you divide it by dt) will be $((a^2+b^2+c^2+d^2) - (e^2+f^2+g^2+h^2))/2$.

We want to prove it is 0, so it is enough to prove a lemma:

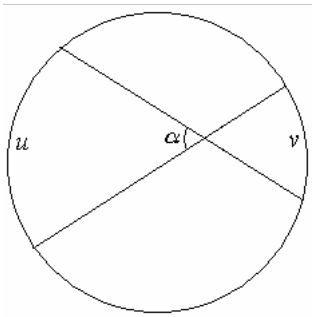
Lemma. Inside a circle, two orthogonal chords cut each other. The length of 4 subintervals which are formed are a, b, c, d. Then $a^2+b^2+c^2+d^2$ doesn't depend on the tilt of this cross, but only on the radius of the circle and the position of the intersection point. Actually, it doesn't even depend on the position of the intersection, and it is always equal to the square of circle's diameter.

Proof of lemma. Let O be the intersection of the chords, and their ends A, B, C, D (named clockwise). By Pythagoras theorem,

$a^2 + b^2 + c^2 + d^2 = OA^2 + OB^2 + OC^2 + OD^2 = AB^2 + CD^2$

Draw a parallel line to AC through D. It will cut the circle in 2 points: D and E.
BDE is a right angle, so BE is a diameter of the circle, so BAE is also right angle.
AE = CD since because of the symmetry, hence we can continue our computation so:
$AB^2 + CD^2 = AB^2 + AE^2 = BE^2 = \text{diameter}^2$
That's it.

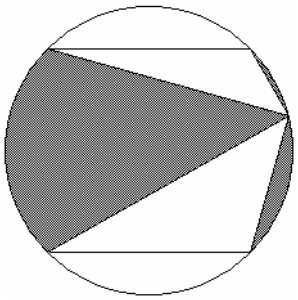
Second solution. Start reducing the circle concentrically, until the circle gets to the intersection. Both dark and light area are being reduced at the same rate. Here we are using:



Lemma. The angle between the chords of a circle is average of two arcs: in the picture on the right $\alpha = (u + v) / 2$.

This is known from elementary geometry, but if you don't know it, take it as an exercise.

So, in our problem, sum of dark and light arcs is the same. When we reduce the radius, at each moment dark are is reduced by *dr* times sum of dark arcs, while light are is reduced by *dr* times sum of light arcs, which is the same.



When the circle contains the intersection of 4 lines, we stop and look. Of the 8 rays, only 4 intersect the boundary, and 4 others look outside the circle. The 4 intersection points with the boundary cut the circle in the vertices of the square. (Since inscribed angles are 45° so intercepted arcs are 90°). So, the circle is split into 5 parts: 3 of one color, and 2 of another. The 2 parts can be divided by chords into 2 segments and 2 triangles. This triangles have bases equal to the side of the square and sum of their heights is the same, so if we will move their common vertex to the center of the circle, their area will be the same. But then, it will definitely become half a circle.

Third solution. (No derivatives, by V. Proizvolov, works for any form with 4 axes of symmetry, taken from the famous Russian "Quant", 1992/10 solutions to quant problems, http://kvant.mirror1.mccme.ru/1992/10/resheniya_zadachnika_kvanta_ma.htm). Would it be a square instead of circle, it would be an easy exercise. So it is enough to prove it for the difference between the bounding square and its circle. Now, just relax and watch:

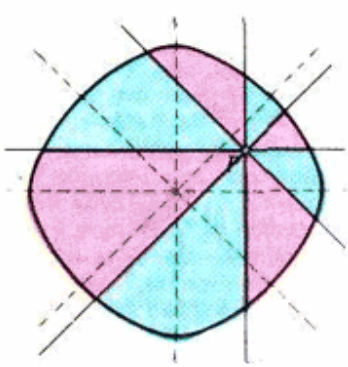


Рис. 1.

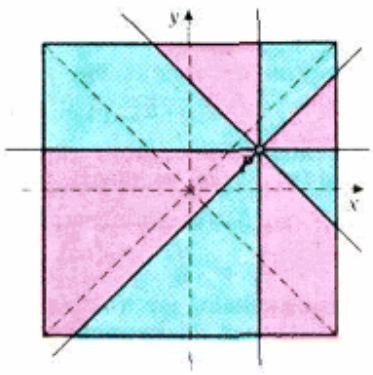


Рис. 2.

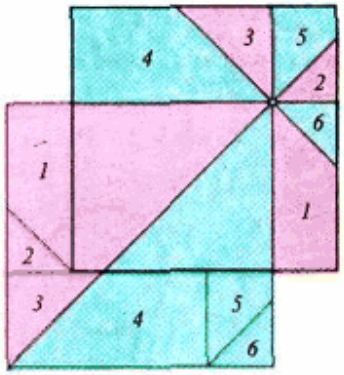
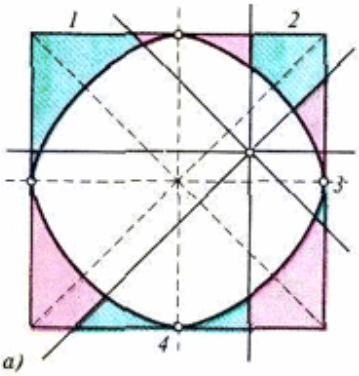
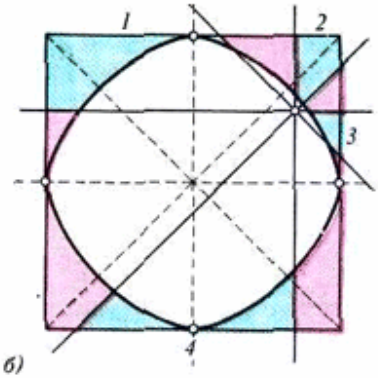


Рис. 3.





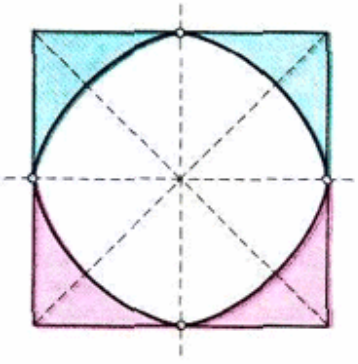


Рис. 4.

Рис. 5. В. Произволов

3. Develop a formula for an area of a polygon, whose vertices are (x_1,y_1) (x_2,y_2), …, (x_n,y_n) (in this order).

First solution.

Let's first develop a formula for triangle.

If the vertices of triangle are (0,0) , (x_1,y_1) , (x_2,y_2) then the area is half the area of the parallelogram, which is determinant, so the answer is $(x_1y_2 - x_2y_1)/2$.

Actually, this is ***oriented area*** – it is usual area when the vertices go clockwise and minus usual area when vertices go counterclockwise. We can get usual are by taking the absolute value, but I prefer area to be oriented for now.

For triangle (x_1,y_1) , (x_2,y_2) , (x_3,y_3): if we shift the triangle by a vector $-(x_1,y_1)$ area won't change, so substitute $(x_2 - x_1, y_2 - y_1)$, $(x_3 - x_1, y_3 - y_1)$ and get:

$((x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1))/2 = ((x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1))/2$

In other words, if You denote S_{ABC} to be the oriented area of triangle ABC then

$$S_{ABC} = S_{OAB} + S_{OBC} + S_{OCA}$$

Here $O = (0,0)$ (For oriented area, unlike the usual area, this formula is always true!).

The formula for quadrilateral is

$S_{ABCD} = S_{ABC} + S_{CDA} = S_{OAB} + S_{OBC} + S_{OCA} + S_{OAC} + S_{OCD} + S_{ODA} = S_{OAB} + S_{OBC} + S_{OCD} + S_{ODA}$

(Since in oriented areas, $S_{OAC} = -S_{OCA}$). This can is generalized for each n by induction, internal terms cancel out, and we get a specific formula:

$$S_{A_1A_2\ldots A_n} = S_{OA_1A_2} + S_{OA_2A_3} + \ldots + S_{OA_nA_1} = \frac{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + \ldots + x_ny_1 - x_1y_n}{2}$$

Remark. One could also find an area of triangle (x_1,y_1) , (x_2,y_2) , (x_3,y_3) directly, by a determinant. Consider pyramid whose vertices are $(x_1,y_1,1)$, $(x_2,y_2,1)$, $(x_3,y_3,1)$, $(0,0,0)$. It is 1/6 of parallelepiped formed by vectors $(x_1,y_1,1)$, $(x_2,y_2,1)$, $(x_3,y_3,1)$.
On the other hand, it is 1/3 of the triangle's area times the height, which is 1.

So the area is $\frac{1}{2}\left|\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}\right|$.

Second solution. Simply take integral of $-ydx$
You will get sum of oriented areas of trapezoids:

$$S_{A_1A_2\ldots A_n} = \frac{(x_1-x_2)(y_1+y_2)+(x_2-x_3)(y_2+y_3)+\ldots+(x_n-x_1)(y_n+y_1)}{2}$$

Which is equivalent to the first solution.

Remark. There is also the third way to write the formula:

$$S_{A_1A_2\ldots A_n} = \frac{x_1(y_2-y_n)+x_2(y_3-y_1)+x_3(y_4-y_2)+\ldots+x_n(y_1-y_{n-1})}{2}$$

Does anybody have a geometric explanation for this one?

4. Two convex polygons P, Q are given, such that Q is strictly inside P.
A point X on the boundary of Q is called good, if there are two points A, B on the boundary of polygon P such that line AB doesn't cut the polygon Q in two parts, and X is a middle point of the interval AB.
Show that there are at least 2 good points.

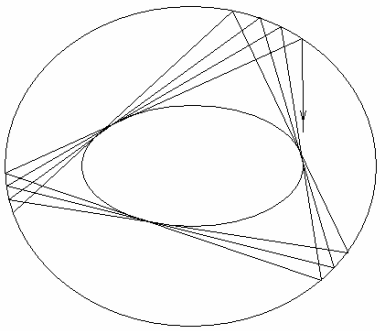
Solution. P is divided by the chord into 2 parts: one contains Q, another doesn't.
The area of the part that doesn't will be called *S*. It can be thought of as a function of the point on the direction of the chord.
When the chord of P is being rotated, while it is still tangent to Q, it sweeps certain area: one part of it sweeps area into *S*, another sweeps area out of *S*.
If a point on Q divides the chord into 2 parts, *x* and *y*, then the growth of *S* is $dS = \frac{x^2-y^2}{2}d\theta$. But then, there should be at least one direction which gives local maximum for S and at least one which gives local minimum. For those two direction, derivative is 0, so $x=y$. QED.

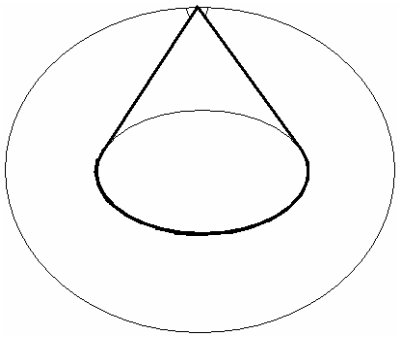
5. Consider a billiard ball on a table bounded by a convex smooth wall without pockets. The ball is so small that we consider it to be a point. The trajectory of the ball is a straight line, until it hits the wall, at the wall it is reflected so that the angle between the wall and the trajectory is preserved. So the trajectory is a broken line, consisting on intervals.

A convex line inside the table is called *caustics*, if the following property holds:

If the trajectory of the billiard ball touches this line once, then it touches it on with interval.
We are given a smooth convex line inside the billiard table which is caustics.
For each point A of the edge of the table, let p(A) be the perimeter of the convex hull of A and the given caustics.
Show that p(A) doesn't depend on A.

Here are pictures demonstrating both equivalent properties.
On the first picture there is a trajectory of the billiard ball, bouncing from the walls and touching the caustics each time.

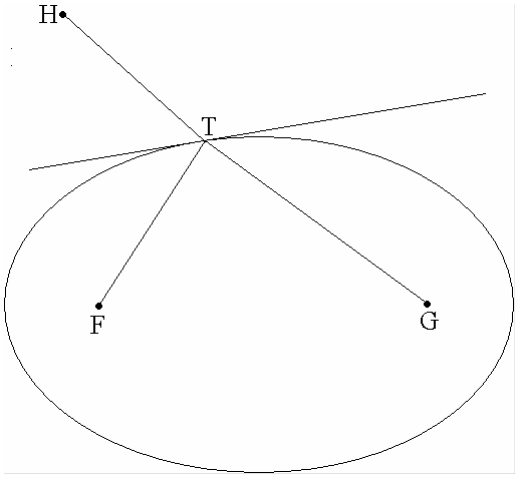




On the second picture, a rope of constant length is thrown around the inner curve, and we draw the outer curve by pulling this rope around with the pen.
The equivalence of those two pictures was noticed in 1972 by Minasian, a student of Yerevan university (and I learned this fact a few mothes ago from a book "Introduction to Ergodic Theory" by a famous mathematician, Jacob Sinai).

I shall be very sketchy, and I am sorry about it, but I am delaying those solutions for a day already.

First proof. Let us pull the rope aside by the pen, and its length will be preserved.
Let us derive this length to see what condition do we get.
The largest part of the rope which winds around the inner curve wont move. The part of it will be release into the straight part of the rope, and taken from another part of the rope, but it will move only slightly (small of order 2) since a tangent is very close to the curve.
The two straight parts will shift, one will become shorter and another longer.
Projection of the straight part of the rope after time *dt* on the same straight part before time *dt* will be of almost the same length as the rope itself (up to order 2) since derivative of cos at 0 is 0. So, the derivative of length will be the same as the distance from the projection of new position of point on the boundary to the straight parts of the rope, from the new position. To make those two distances cancel, the trajectory should be outer bisector for the two straight parts of the rope.
But the trajectory is the outer bisector, if and only if a billiard ball could be going along this rope.

Second solution. First proof it for a simple special case: when the inner part is an interval. This thing is called "the optical property of the ellipse".

The ellipse is a curve, which is a locus of points such that the sum of two distances from the point to two given points, called foci (plural of focus), is constant.
The optical property of the ellipse is that the billiard ball, arriving from the first focus, returns to the second focus after it hits the wall of elliptical billiard table.
This is precisely the special case of our problem.
After we have proven it, we can consider a more general special case, precisely when the inner curve is a convex polygon. It follows directly from the optical property of the ellipse (when you stretch the rope with a pen around the polygon, you get that an outer curve consists of elliptical arcs, and everything follows).
After we prove it for polygons, we simply approximate any convex by a polygon, and obtain the general theorem from the specific case.
The only nontrivial part of this proof is the optical property of the ellipse, other parts are easy. So, for those of you who never learned it, here is the proof of the optical property:
Denote the foci F, G, a point on the ellipse T, draw a tangent to the ellipse at T.
The reflection of F with respect to that tangent line is H. All we need to prove is that points G, T, H are on one line.
Assume the opposite: segment GH doesn't contain T, it cuts the tangent line in a different point, S. Then TH+TG > SH+SG. But, due to symmetry, TF=TH, SH=SF, so TF+TG > SF+SG. But it cant be, since ellipse is convex (why?), so S is outside the ellipse, so TF+TG < SF+SG.