These problems are about determinants and linear algebra.

1. Compute $\left\|\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}\right\|$

(First row/column consist of 1's, any number which is neither in the first row nor in the first column, is sum of its neighbors from above and from the left).

Solution. This determinant is one, it is shown by some cunning version of Gauss method. Det is not changed when You subtract one line from another. So, subtract line 4 from line 5, then line 5 becomes line shifted left by 1, precisely: 0 1 5 15 35.
Now subtract line 3 from line 4, line 4 will become 0 1 4 10 20
Now subtract line 2 from line 3, line 3 it also shifts by 1.
Then subtract line 1 from line 2. What You get is:

$$\left\|\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 6 & 10 \\ 0 & 1 & 4 & 10 & 20 \\ 0 & 1 & 5 & 15 & 35 \end{pmatrix}\right\|$$

Subtracting one column from another also keeps determinant.
So, subtract column 4 from column 5. Column 5 moves down.
Subtract column 3 from column 4. Column 4 moves down.
Subtract column 2 from column 3. Column 3 moves down.
Subtract column 1 from column 2. Column 2 moves down. Now we get:

$$\left\|\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 6 & 10 \\ 0 & 1 & 4 & 10 & 20 \end{pmatrix}\right\|$$

Now again, perform similar actions on rows and then on columns, three times.
In the end, you will get the unit matrix. And det is still the same.

2. Consider two quadratic polynomials, ax^2+bx+c, dx^2+ex+f, where $a,d\neq 0$. Prove that they have the common root if and only if the matrix

$$\begin{pmatrix} a & b & c & 0 \\ 0 & a & b & c \\ d & e & f & 0 \\ 0 & d & e & f \end{pmatrix}$$

is degenerate.

Proof. Suppose they do have a common root α. Then multiplying this matrix by $\begin{pmatrix} \alpha^3 \\ \alpha^2 \\ \alpha \\ 1 \end{pmatrix}$ will give You 0 vector. So, if we have a common root, than a matrix is degenerate. We need to prove the other direction also.

Denote the 2 (complex) roots of the first polynomial are x_1, x_2 and of the second polynomial y_1, y_2. Then by Vieta theorem we get:

$$\left|\begin{pmatrix} a & b & c & 0 \\ 0 & a & b & c \\ d & e & f & 0 \\ 0 & d & e & f \end{pmatrix}\right| = \left|\begin{pmatrix} a & a(x_1+x_2) & ax_1x_2 & 0 \\ 0 & a & a(x_1+x_2) & ax_1x_2 \\ d & d(y_1+y_2) & dy_1y_2 & 0 \\ 0 & d & d(y_1+y_2) & dy_1y_2 \end{pmatrix}\right| =$$

$$= a^2d^2\left|\begin{pmatrix} 1 & x_1+x_2 & x_1x_2 & 0 \\ 0 & 1 & x_1+x_2 & x_1x_2 \\ 1 & y_1+y_2 & y_1y_2 & 0 \\ 0 & 1 & y_1+y_2 & y_1y_2 \end{pmatrix}\right|$$

If You sum up all ways to place 4 rooks on this matrix, and You can express this determinant as a polynomial in x_1, x_2, y_1, y_2.

The degree of the polynomial is 4 (why?) and it is divisible by $(y_1-x_1)\cdot(y_2-x_1)\cdot(y_1-x_2)\cdot(y_2-x_2)$ because if some polynomial of several variables is a constant zero on the zeroes of some linear polynomial, such as y_1-x_1, then it is divisible by that linear polynomial (and why am I so sure of this?).

So, the polynomial I get is $C(y_1-x_1)\cdot(y_2-x_1)\cdot(y_1-x_2)\cdot(y_2-x_2)$ where *C* is a polynomial of degree 0, i. e. a number.

Notice that both the determinant and $(y_1 - x_1)\cdot(y_2 - x_1)\cdot(y_1 - x_2)\cdot(y_2 - x_2)$ contain a monomial $y_1^2 y_2^2$ so $C = 1$.
That's it.

Remarks. (1) In precisely the same way You can see that the determinant of

$$\begin{vmatrix} a_n & & \dots & a_1 & a_0 & & \\ & \ddots & & & \ddots & \ddots & \\ & & a_n & & \dots & a_1 & a_0 \\ b_m & \dots & b_1 & b_0 & & & \\ & b_m & \dots & b_1 & b_0 & & \\ & & \ddots & & \ddots & \ddots & \\ & & & b_m & \dots & b_1 & b_0 \end{vmatrix}$$

(first parallelogram consists of *m* lines, second of *n* lines), is 0 iff the polynomials $a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0, b_m x^m + \ldots + b_2 x^2 + b_1 x + b_0$ have a common root.
I didn't write it in the generic form from all the time for two obvious reasons.
Firstly, I was lazy to draw complicated determinants; secondly, You might prefer to do the general case by Yourself.

(2) This determinant is called resultant of two polynomials.

(3) The resultant of a polynomial with itself (sometimes divided by the first coefficient) is called a discriminant of the polynomial. You probably learned it in kita het for quadratic polynomials.

3. A quadric in plane is a locus of zeroes of an equation of order 2:
$ax^2 + bxy + cy^2 + dx + ey + f = 0$. (At least one coefficient should be nonzero)
a. Show that for each 5 points there exists a quadric containing all of them.
b. Show that if two quadrics have only finite number of common points, then it isn't bigger than 4.

a. A condition that given quadric has a certain point imposes a linear condition on its six coefficients (given the point) 5 points give us 5 linear conditions. 6 variables, 5 linear condition, homogenous system ➔ it must have a nontrivial solution.

b. First solution. Assume we have two quadrics which have more than 4, but a finite number of intersections. So, there are finite number of lines passing through 2 intersections, and we can rotate this picture so that none of those lines will be horizontal.

After rotation quadrics will remain quadrics (since the formulas for the rotation are linear). Let us say that their equations will be:

$$a_1x^2+\left(b_1y+d_1\right)x+\left(c_1y^2+e_1y+f_1\right)=0$$

$$a_2x^2+\left(b_2y+d_2\right)x+\left(c_2y^2+e_2y+f_2\right)=0$$

To have an intersection on the level of certain y means to have a common root x for these two polynomials. Which means there will be a zero resultant for this y :

$$0=\begin{vmatrix} a_1 & b_1y+d_1 & c_1y^2+e_1y+f & 0 \\ 0 & a_1 & b_1y+d_1 & c_1y^2+e_1y+f \\ a_2 & b_2y+d_2 & c_2y^2+e_2y+f & 0 \\ 0 & a_2 & b_2y+d_2 & c_2y^2+e_2y+f \end{vmatrix}$$

Number of root of this resultant, which is polynomial in y, is a number of intersections. But we see it is polynomial of degree no more than 4.

Remark. This proof is easily generalizable, and we get a theorem about any two algebraic curves: if their degrees are M and N and they have only finite number of common points, then it is not bigger than M·N. This fact is called Bezout theorem.
Here comes a second solution, which is more elementary, but it is not generalizable (as far as I know).

Second solution. In analytic geometry there is a classification of quadrics: nondegenerate quadric is either non-degenerate (an ellipse, a parabola, a hyperbola) or degenerate (union of two lines, a line, an isolated point, an empty set).
Degenerate cases are easily verified. For example, a line can have no more than 2 intersections with quadric, since it is a solution of quadratic equation.
So, two lines can have no more than 4 intersections with a quadric.

Hence, it is sufficient to consider the case in which both quadrics are non-degenerate.
Choose a tangent to the first quadric at some point which is not their intersection.
Perform a projection of this plane which will send this tangent to infinity.
(By the way, why does a projection send quadrics into quadrics???)
The first quadric will become a parabola.
After some stretching and rotating, the first quadric will become $y=x^2$, so it will have only one y for each x. The second quadric will be
$ax^2+bxy+cy^2+dx+ey+f=0$. Substitute $y=x^2$, You get a polynomial of degree 4. so only 4 values of x allow intersection, and for each x there is just one y.

4. Find the roots of the polynomials:

a. $\begin{vmatrix} x & 1 & 0 & 0 & 1 \\ 1 & x & 1 & 0 & 0 \\ 0 & 1 & x & 1 & 0 \\ 0 & 0 & 1 & x & 1 \\ 1 & 0 & 0 & 1 & x \end{vmatrix}$ b. $\begin{vmatrix} x & 1 & 0 & 0 & 0 \\ 1 & x & 1 & 0 & 0 \\ 0 & 1 & x & 1 & 0 \\ 0 & 0 & 1 & x & 1 \\ 0 & 0 & 0 & 1 & x \end{vmatrix}$

a. Solution. Consider a matrix $R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

It is rotating the standard basis. We can guess its eigenvectors.

Let $\xi = e^{2\pi i/5}$ ("a root of degree 5 of 1"). Then the vector $\begin{pmatrix} 1 \\ \xi^k \\ \xi^{2k} \\ \xi^{3k} \\ \xi^{4k} \end{pmatrix}$, when You multiply R

by it, is multiplied by ξ^k. For k=0, 1, 2, 3, 4 we get in this way 5 different eigenvectors and 5 different eigenvalues. Eigenvectors corresponding to different eigenvalues are linearly independent. Let switch to this eigenbasis.

Our original matrix is actually $R+R^{-1}+x\mathrm{E}$, where E is a unit matrix. So, in eigenbasis,

R= $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 & 0 \\ 0 & 0 & \xi^2 & 0 & 0 \\ 0 & 0 & 0 & \xi^3 & 0 \\ 0 & 0 & 0 & 0 & \xi^4 \end{pmatrix}$ so the original matrix is

$$\begin{pmatrix} x+2 & 0 & 0 & 0 & 0 \\ 0 & x+\xi^4+\xi & 0 & 0 & 0 \\ 0 & 0 & x+\xi^3+\xi^2 & 0 & 0 \\ 0 & 0 & 0 & x+\xi^2+\xi^3 & 0 \\ 0 & 0 & 0 & 0 & x+\xi^4+\xi \end{pmatrix},$$

and its determinant is

$$(x+2)\left(x+\xi^4+\xi\right)\left(x+\xi^3+\xi^2\right)\left(x+\xi^2+\xi^3\right)\left(x+\xi^4+\xi\right)=$$

$$(x+2)\left(x+2\cos\left(\frac{2\pi}{5}\right)\right)^2\left(x+2\cos\left(\frac{4\pi}{5}\right)\right)^2$$

And roots are $-2,-2\cos\left(\frac{2\pi}{5}\right),-2\cos\left(\frac{4\pi}{5}\right)$ (the last two are double roots).

Remark. This matrix R is a well know mathematical object, and its eigenbasis is even more famous. Passing to this basis is called discreet Fourier Transform, and it has a lot of magical properties (did You ever hear, for example, about multiplying numbers of length N in O(N·logN) operations?).

b. Here again, we shall guess the eigenvector. The key to guessing is a nice trigonometric formula: sin x + siny.
A shall allow myself to show You its proof (not only because in some schools they don't prove formulas, but also because I have a special prove which hints the solution).
Take two unit vector (cos *x*, sin *x*) and (cos *y*, sin *y*) and sum them. We get a rombus. The angle of the sum vector is *(x+y)/2* since it is a bisector. The length of the diagonal of the rhombus (מעוין) is 2cos(*(x-y)/2*). So,
cos *x* + cos y = $2\cos((x-y)/2)\cos((x+y)/2)$
sin *x* + sin y = $2\cos((x-y)/2)\sin((x+y)/2)$

So, we have

$$\begin{pmatrix} x & 1 & 0 & 0 & 0 \\ 1 & x & 1 & 0 & 0 \\ 0 & 1 & x & 1 & 0 \\ 0 & 0 & 1 & x & 1 \\ 0 & 0 & 0 & 1 & x \end{pmatrix}\begin{pmatrix} \sin(k\pi/6) \\ \sin(2k\pi/6) \\ \sin(3k\pi/6) \\ \sin(4k\pi/6) \\ \sin(5k\pi/6) \end{pmatrix}=\left(x+2\cos(k\pi/6)\right)\begin{pmatrix} \sin(k\pi/6) \\ \sin(2k\pi/6) \\ \sin(3k\pi/6) \\ \sin(4k\pi/6) \\ \sin(5k\pi/6) \end{pmatrix}$$

For k = 1, 2, 3, 4, 5. And since 5×5 matrix can have only 5 distinct eigenvalues, the only answers are $-2\cos(k\pi/6)$ for 1, 2, 3, 4, 5.

Remark. Of course, all this can be said for each N and not just for 5.
The polynomial related to trigonometry were have a lot of nice properties, some of them were studied by Chebyshev and bear his name (the particular polynomials we showed are not Chebyshev's, but are closely related to them).

5. Write an equation which holds if and only if the four points (x_1,y_1), (x_2,y_2), (x_3,y_3) (x_4,y_4) lie on one circle or one straight line.

First solution

An equation of a line or a circle is of a form $a(x^2 + y^2) + bx + cy + d = 0$

And it should hold for all 4 points, so the condition is that the matrix

$$\begin{pmatrix} x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \\ x_4^2+y_4^2 & x_4 & y_4 & 1 \end{pmatrix}$$

is degenerate. So a condition is a determinant of this matrix = 0. That's it.

Remarks. Denote those points A, B, C, D and O the zero. Then this formula has some geometric meaning. Develop it with respect to the first column, you get

$$0 = \begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \\ x_4^2+y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = \left(x_1^2+y_1^2\right)\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - \left(x_2^2+y_2^2\right)\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} +$$

$$+\left(x_3^2+y_3^2\right)\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - \left(x_4^2+y_4^2\right)\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

But those 3×3 determinants have an obvious meaning as twice the area of the triangle, (the area of the triangle which is oriented, i. e. has minus sign iff its coordinates are mentioned clockwise) so our condition takes form

$$OA^2 \cdot S_{BCD} - OB^2 \cdot S_{ACD} + OC^2 \cdot S_{ABD} - OD^2 \cdot S_{ABC} = 0$$

So, we get a geometric theorem – ABCD is inscribed, iff for any point O the above condition holds. If O is the center of the circle, OA=OB=OC=OD=R, we get a trivial condition: $S_{BCD} + S_{ABD} = S_{ACD} + S_{ABC}$.

On the contrary, if O=A we get $AB^2 \cdot S_{ACD} + AD^2 \cdot S_{ABC} = AC^2 \cdot S_{ABD}$

If ABCD is really inscribed, all triangles are inscribed in the same circle, so each area is are product of their sides divided by 4R. So, if we multiply by 4R we get:

$$AB^2 \cdot AC \cdot CD \cdot DA + AD^2 \cdot AB \cdot BC \cdot CA = AC^2 \cdot AB \cdot BD \cdot DA$$

Divide it by $AB \cdot AC \cdot AD$ and You get the famous Ptolemy's theorem, which holds for every inscribed quadrilateral: $AB \cdot CD + AD \cdot BC = AC \cdot BD$.

Second solution.

Consider points A, B, C, D as complex numbers.

Then the argument of a complex number (A-B)/(C-B) is precisely the angle which is needed to rotate a vector a vector BA to the direction of vector BC.
The argument of complex number (A-D)/(C-D) is precisely the angle which is needed to rotate a vector a vector DA to the direction of vector DC.

Those two angles should be either equal (if B, D are on the same side of line AC) or should be opposite (if they are on different sides), so anyway, the condition is that the ratio of the two above ratios $\frac{A-B}{C-B}:\frac{A-D}{C-D}=\frac{(A-B)(C-D)}{(C-B)(A-D)}$ is real.

Remark. This ratio is famous in projective geometry and it is called cross ratio.
It is famous for being an invariant of projective transformations of projective line, and if You consider complex projective line, $\mathbf{CP^1}$ (if You know such words) our conclusions become quite obvious.

So, let us multiply nominator and denominator by the conjugate of denominator,

$$\frac{(A-B)(C-D)\overline{(C-B)(A-D)}}{real}=real$$

And the condition is: $\operatorname{Im}\left((A-B)(C-D)\overline{(C-B)(A-D)}\right)=0$.

Now we can tight it in coordinates, if need be.

$$0=\operatorname{Im}\left(\left((x_1-x_2)+i(y_1-y_2)\right)\left((x_3-x_4)+i(y_3-y_4)\right)+\right.$$
$$\left.+\left((x_3-x_2)-i(y_3-y_2)\right)\left((x_1-x_4)-i(y_1-y_4)\right)\right)=$$
$$=\operatorname{Im}\left(\left(\left((x_1-x_2)(x_3-x_4)-(y_1-y_2)(y_3-y_4)\right)+\right.\right.$$
$$\left.+i\left((x_1-x_2)(y_3-y_4)+(y_1-y_2)(x_3-x_4)\right)\right)\times$$
$$\times\left(\left((x_3-x_2)(x_1-x_4)-(y_3-y_2)(y_1-y_4)\right)-\right.$$
$$\left.\left.-i\left((x_3-x_2)(y_1-y_4)+(y_3-y_2)(x_1-x_4)\right)\right)\right)=$$
$$=-\left((x_1-x_2)(x_3-x_4)-(y_1-y_2)(y_3-y_4)\right)$$
$$\left((x_3-x_2)(y_1-y_4)+(y_3-y_2)(x_1-x_4)\right)+$$
$$+\left((x_1-x_2)(y_3-y_4)+(y_1-y_2)(x_3-x_4)\right)$$
$$\left((x_3-x_2)(x_1-x_4)-(y_3-y_2)(y_1-y_4)\right)$$

Well, if I didn't make a mistake in the computation.
Anyway, like in first solution, it is a polynomial of order 4 and it is 0 when 2 points coincide.