## Targil 14 – functions.

**1.** Let $f:\mathbb{R}\to\mathbb{R}$ be thrice differentiable.

Show that there exists $\xi\in(-1,1)$ s. t. $\dfrac{f'''(\xi)}{6}=\dfrac{f(1)-f(-1)}{2}-f'(0)$.

Proof. One proof, which is not completely honest, but acceptable at IMC, is to say that the LHS is $f[-1,0,0,1]$ (Newton's divided difference), and hence it is true.

$$\begin{array}{llll} f(-1) & f(0)-f(-1) & f'(0)-\big(f(0)-f(-1)\big) & \dfrac{f(1)-2f'(0)-f(-1)}{2} \\ f(0) & f'(0) & f(1)-f(0)-f'(0) & \\ f(0) & f(1)-f(0) & & \\ f(1) & & & \end{array}$$

A more honest way is to give a proof, at least for this special case. Construct a polynomial of degree 3 which coincides with $f$ at points –1, 0, 1, and touches $f$ at 0.

$$g(x)=-\frac{f(-1)}{2}x^2(x-1)-f(0)(x-1)(x+1)+\frac{f(1)}{2}x^2(x+1)-f'(0)x(x-1)(x+1)$$

So, the function $f-g$ has 3 root, one of those of multiplicity 2, on [–1, 1].

By Rolle's theorem, its third derivative has at least one root. So, at one point, the third derivative of $f$ coincides with the third derivative of $g$. But the third derivative of $g$ is constant and equal to LHS of the original expression times 6.

**2.** Does there exist a continuously differentiable function s. t. for every real $x$ $f(x)>0$ and $f'(x)>f\big(f(x)\big)$ ?

**Solution.** $f$ is positive, hence $f'$ is positive, hence $f$ is growing monotonically. Hence $f'(x)>f\big(f(x)\big)>f\big(f(0)\big)=const>0$, hence the slope of $f$ is bounded from below, hence when we go to negative direction the value of the function goes down by at least given rate, hence it reaches negative values, which was specifically forbidden.

A more algebraic way to write this down (and it would cause less argument between the judges at the Olympiad):
Denote $C = f(0) > 0$.

$f'(x) > f(f(x)) > 0$, hence f monotonically increases.

$f(x) > 0$, therefore $f(f(x)) > f(0) = C$.

So $f'(x) > f(f(x)) > C$.

By Lagrange for each $x < y$ there is $z$ between them such that

$f(y) - f(x) = f'(z)(y-x) > C(y-x)$

$f(y) - C(y-x) > f(x)$

Choose any $y$, take $\Delta = \dfrac{2f(y)}{C}$ and take $x = y - \Delta$.

Then $0 > f(y) - C \cdot \Delta = f(y) - C(y-x) > f(x)$.

Contradiction.

**3.** Consider continuous function $f:[0,1] \to \mathbb{R}$ such that

$$x \cdot f(y) + y \cdot f(x) \le 1.$$

a) Prove that $\int_0^1 f(x)\,dx \le \dfrac{\pi}{4}$.

b) Find such a function f that the inequality of a) will become equality.

**Solution. a)**

$$\int_0^1 f(x)\,dx = \int_0^{\pi/2} f(\sin t)\,d\sin t = \int_0^{\pi/2} f(\sin t)\cos t \cdot dt = \int_0^{\pi/2} f(\cos u)\sin u \cdot du$$

Here we did substitutions $x = \sin t$ , $u = \pi/2 - t$ .

Hence the integral can be computed as the mean of the two last expressions:

$$\int_0^1 f(x)\,dx = \frac{1}{2}\left( \int_0^{\pi/2} f(\sin t)\cos t \cdot dt + \int_0^{\pi/2} f(\cos u)\sin u \cdot du \right) =$$

$$= \frac{1}{2}\int_0^{\pi/2} \left( f(\sin t)\cos t + f(\cos t)\sin t \right) \cdot dt \le \frac{1}{2}\int_0^{\pi/2} 1 \cdot dt = \frac{\pi}{4}$$

**b)** For example: $f(x)=\sqrt{1-x^2}$ . The integral is $\frac{\pi}{4}$ since the area bounded by the graph is a quarter of unit circle. The condition holds
$x\cdot f(y)+y\cdot f(x)=x\cdot\sqrt{1-y^2}+\sqrt{1-x^2}\cdot y\le\left(x^2+1-x^2\right)\left(1-y^2+y^2\right)=1$
by Cauchy-Schwartz inequality.

It can be also shown with trigonometric substitution: $f(x)=\cos(\arcsin x)$,
$x=\sin\alpha, y=\sin\beta, f(x)=\cos\alpha, f(y)=\cos\beta$ therefore:
$x\cdot f(y)+y\cdot f(x)=\sin\alpha\cos\beta+\sin\beta\cos\alpha=\sin(\alpha+\beta)\le 1$

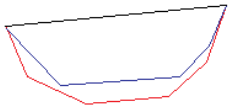
**4.** Let $f,g:[a,b]\to(0,\infty)$ be continuous, non-decreasing functions, such that for every $x\in[a,b]$ we have $\int_a^x\sqrt{f(t)}dt\le\int_a^x\sqrt{g(t)}dt$

and $$\int_a^b\sqrt{f(t)}dt=\int_a^b\sqrt{g(t)}dt$$

Prove that $$\int_a^b\sqrt{1+f(t)}dt\ge\int_a^b\sqrt{1+g(t)}dt$$

**Solution.** $\sqrt{f(t)},\sqrt{g(t)}$ , are non-decreasing, therefore
$F(x)=\int_a^x\sqrt{f(t)}dt$ , $G(x)=\int_a^x\sqrt{g(t)}dt$ are convex. So, we have two convex graphs one above another, with common ends: $F(a)=0=G(a)$ ,
$F(b)=\int_a^b\sqrt{f(t)}dt=\int_a^b\sqrt{g(t)}dt=G(b)$.
Let us rewrite the statement we have to prove in terms of *F*, *G*:
$\int_a^b\sqrt{1+\left(F'(t)\right)^2}dt\ge\int_a^b\sqrt{1+\left(G'(t)\right)^2}dt$ .
That is precisely the expression for the length of the curve:
$\int_a^b\sqrt{1+\left(F'(x)\right)^2}dx=\int_a^b\sqrt{1+\left(\frac{dF}{dx}\right)^2}dx=\int_a^b\sqrt{dx^2+dF^2}=\int_a^b\sqrt{dl^2}=\int_a^b dl=length$.

So, what we have to prove is that the graph of $F$ is longer than the graph of G. The length of arc of a convex function might be computed as a limit of inscribed broken lines. So all we need to prove is



**Lemma.** If one convex polygon is inside another, then its perimeter is smaller.

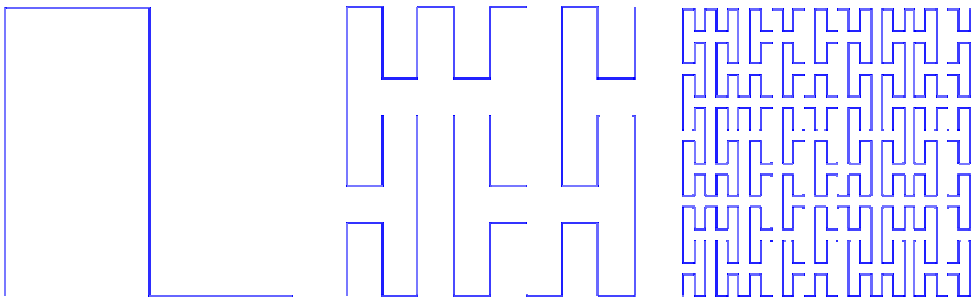
**Proof.** Extend one of the first interval of the internal line. It will cut the external polygon into 2 parts. If we drop the lower part, the perimeter of the external curve decreases. Now we may omit the common side of two curves which we have created, and reduce the number of the sides of the upper line, so upper line is shorter by induction.

**5*.** Does there exist a

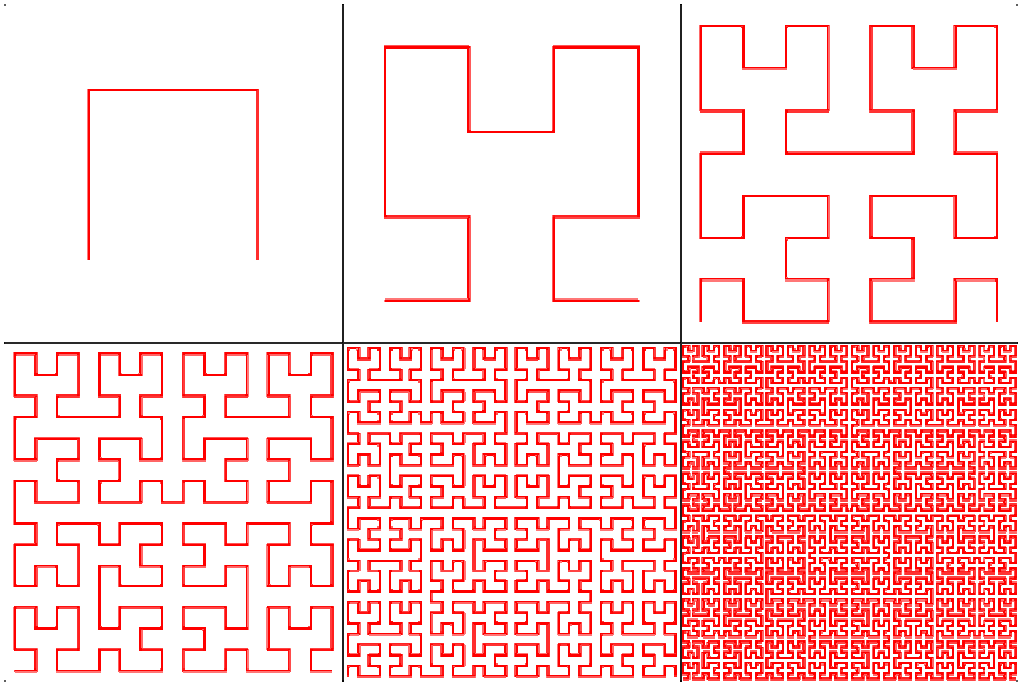
a) continuous
b) monotone
c) continuously differentiable

function $f:[0,1]\to[0,1]$ s. t. every $y\in[0,1]$ it has uncountable number of inverse images of $y$ ?
Uncountable means strictly bigger than $\aleph_0$.
Inverse image of $y$ is such an $x$ that $f(x) = y$.

**Solution. a)** Consider Peano curve, which is a continuous curve covering the square, or (which is the same) an onto map from [0,1] to [0,1]×[0,1].
The first coordinate of Peano curve is a continuous function, and inverse image has at least one point for each possible value of second coordinate, which is $2^{\aleph_0}$.
Peano curve is constructed by limiting procedure like this:
http://en.wikipedia.org/wiki/Space-filling_curve



Here is another example by Hilbert:



**b)** No. Inverse image of a point under monotone function is either a point or an interval (if two points belong to inverse image, than all intermediate points also do). Each interval contains a rational point.
If we would have such a function, then inverse image of each point would have a rational point. That would give an injective (חח"ע) mapping from [0,1] to the countable set of rational numbers, which can't exist.

**c)** No. Let $f$ be such a map. Then for each value $y$ of this map there is an $x$ such that $y = f(x)$ and $f'(x) = 0$, since inverse image of $y$ contains an accumulation point and clearly derivative at the accumulation point should be 0.
Every $x$ such that $f'(x) = 0$ belongs to an open interval on which $|f'(x)| \leq \varepsilon$. Union of those intervals for all $y$'s can be represented as disjoint union of intervals. Total length of those intervals $\leq 1$ , so f will send union of those intervals to the union of intervals of total length $\leq \varepsilon$.
If $\varepsilon < 1$, then we see that image of these intervals can't cover everything, QED.
**Remark.** If we choose smaller and smaller ε, we see that the image of these points such that $f'(x) = 0$ is of measure 0. This is actually the 1-simensional case of the famous Sard's theorem.

Definitions. (1) A function $f$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called ***differentiable***, if it can be locally, at each point, approximated by linear function $l(x) = Ax + b$,

where A and $b$ are matrix and vector of appropriate size (function depends on the point at which we approximate).
(2) Linear transformation, corresponding to A matrix, which approximates differentiable function at a point, is called ***differential*** of a function.
(3) If rank(differential at point $x$) < dimension(target space) then $x$ is called ***critical point***.
(4) If $x$ is critical point, $f(x)$ is called ***critical value***.

**Theorem (Sard).** The set of critical values is of measure 0.

**6**.** For a continuous function $f: \mathbb{R} \to \mathbb{R}$ it is given that any positive

real $x$, $y$ the sequence $f(x+ny)$ , for $n \in \mathbb{N}$, tends to infinity.

Does it follow that $f(x) \to \infty$ as $x \to \infty$ ?

It is similar to the problem we had in the competition, but now we require continuity and the answer is different.

**Solution.** Yes, it does.
Suppose it doesn't. Then for some $M$ there is a sequence $x_k$, converging to infinity, such that $f(x_k) < M$. Then, since $f$ is continuous, for $|x - x_k| < \varepsilon_k$ , we have $f(x_k) < 2M = N$ , where $\varepsilon_k$ are small numbers, chosen separately for different $k$. So, to get the contradiction we need to do one thing: build an arithmetic sequence which intersects infinite subset of these small intervals.
Assume we have built a sequence $\{ny\}$ which intersects $K$ intervals:
$n_k y \in \left(x_{m_k} - \varepsilon_{m_k}, x_{m_k} + \varepsilon_{m_k}\right)$, for some indices $n_k$ , $m_k$ for $k \le K$.
We can move the y in certain interval so that the conditions
$n_k y \in \left(x_{m_k} - \varepsilon_{m_k}, x_{m_k} + \varepsilon_{m_k}\right)$ still hold for the same $n_k$ , $m_k$, because intersection of open intervals is still an open interval, if it is nonempty.
We shall find such $y$ that satisfies this condition for as large $K$ as we want by induction over $K$. For $K = 1$ it is obvious.
Assume that $n_k y \in \left(x_{m_k} - \varepsilon_{m_k}, x_{m_k} + \varepsilon_{m_k}\right)$, for given $n_k$ , $m_k$ for $k \le K$, in the interval $y \in (\alpha Y, Y)$, where $\alpha < 1$.
Possible values of $ny$ will cover the interval between $ny$ and $(n+1)y$ if $(n+1)\alpha Y < nY$ i. e. $1 + \frac{1}{n} = \frac{n+1}{n} < \frac{1}{\alpha}$, or $n > \frac{\alpha}{1-\alpha}$.

So, all numbers above $Y\frac{\alpha}{\alpha-1}$ will be covered by possible values of $ny$.
But $x_k$ tends to infinity, so we can choose $x_m$ such that it can be equal to $ny$ for a certain value of $y$ in the interval. This completes the induction.
By this inductive procedure we shall build an infinite set of indices $m_k$ and a nested system of intervals $(\alpha_k Y_k, Y_k)$ such that if we choose $y$ in interval number $K$ then $\{ny\}$ intersects intervals $\left(x_{m_k}-\varepsilon_{m_k}, x_{m_k}+\varepsilon_{m_k}\right)$ for $k \le K$, the intersection of all those intervals has at least on point $y$, and for that $y$ sequence $\{ny\}$ intersects infinite number of intervals. Hence $f(ny)$ doesn't tend to infinity, contradiction, QED.