Targil 14 – functions.

1. Let $f:\mathbb{R}\to\mathbb{R}$ be thrice differentiable.

Show that there exists $\xi\in(-1,1)$ s. t. $\dfrac{f'''(\xi)}{6}=\dfrac{f(1)-f(-1)}{2}-f'(0)$.

2. Does there exist a continuously differentiable function s. t. for every real x $f(x)>0$ and $f'(x)>f\left(f(x)\right)$?

3. Consider continuous function $f:[0,1]\to\mathbb{R}$ such that

$$x\cdot f(y)+y\cdot f(x)\le 1.$$

a) Prove that $\int_0^1 f(x)\,dx\le\dfrac{\pi}{4}$.

b) Find such a function f that the inequality of a) will become equality.

4. Let $f,g:[a,b]\to(0,\infty)$ be continuous, non-decreasing functions, such that for every $x\in[a,b]$ we have $\int_a^x\sqrt{f(t)}dt\le\int_a^x\sqrt{g(t)}dt$

and $$\int_a^b\sqrt{f(t)}dt=\int_a^b\sqrt{g(t)}dt$$

Prove that $$\int_a^b\sqrt{1+f(t)}dt\ge\int_a^b\sqrt{1+g(t)}dt$$

5*. Does there exist a

a) continuous
b) monotone
c) continuously differentiable

function $f:[0,1]\to[0,1]$ s. t. every $y\in[0,1]$ it has uncountable number of inverse images of y ?

Uncountable means strictly bigger than $\aleph_0$.

Inverse image of y is such an x that $f(x) = y$.

6.** For a continuous function $f:\mathbb{R}\to\mathbb{R}$ it is given that any positive real x, y the sequence $f(x+ny)$, for $n\in\mathbb{N}$, tends to infinity.

Does it follow that $f(x)\to\infty$ as $x\to\infty$?

(It is similar to the problem we had in the competition, but now we require continuity).