Targil 13 – limits.

1. Compute $\lim_{x\to 0+}\int_x^{2x}\frac{\sin^m(t)}{t^n}dt$, where m and n are natural numbers,
(the answer may depend on m and n).

Solution. By mean value theorem of integration
$\int_x^{2x}\frac{\sin^m(t)}{t^n}dt=\int_x^{2x}\left(\frac{\sin(t)}{t}\right)^m\frac{1}{t^{n-m}}dt=\left(\frac{\sin(y)}{y}\right)^m\int_x^{2x}\frac{1}{t^{n-m}}dt$, where $x\le y\le 2x$.
Hence $\frac{\sin(y)}{y}\xrightarrow[x\to 0]{}1$, so $\left(\frac{\sin(y)}{y}\right)^m\xrightarrow[x\to 0]{}1$. Therefore
$$\lim_{x\to 0+}\int_x^{2x}\frac{\sin^m(t)}{t^n}dt=\lim_{x\to 0+}\int_x^{2x}\frac{1}{t^{n-m}}dt=\frac{m-n}{t^{n-m+1}}\Big|_x^{2x}=\lim_{x\to 0+}(m-n)\left(\frac{1}{2^{n-m+1}}-1\right)\frac{1}{x^{n-m+1}}$$
if $n-m\neq -1$. Since the integral is positive anyway, it only matters whether $\lim_{x\to 0+}\frac{1}{x^{n-m+1}}$ tends to zero or to infinity in these cases, so
if $n-m>-1$ then it tends to infinity,
if $n-m<-1$ then it tends to zero,
and if $n-m=-1$ then
$$\lim_{x\to 0+}\int_x^{2x}\frac{\sin^m(t)}{t^n}dt=\lim_{x\to 0+}\int_x^{2x}\frac{1}{t}dt=\lim_{x\to 0+}\left(\ln(2x)-\ln x\right)=\ln 2$$
Remark. This problem is pretty obvious, but it really did appear on IMC.

2. a) Does $\sum_{k=100}^{\infty}\frac{1}{k\cdot(\ln k)\cdot(\ln(\ln k))}$ converge?

b) Does $\sum_{k=100}^{\infty}\frac{1}{k\cdot(\ln k)\cdot(\ln(\ln k))^2}$ converge?

Solution. a) Apply the "harmonic trick" : split the sequence into sub-strips: strip number n will contain numbers between 2^n+1 and $2n$+1.
Then all numbers in the strip number n are (up to a bounded factor) $\frac{1}{2^n\cdot n\cdot(\ln n)}$ and there are 2^n elements in this strip.

Therefore the sum of that strip is $\frac{1}{n\cdot(\ln n)}$.

So it is enough to investigate the convergence of the series $\sum_{n=?}^{\infty}\frac{1}{n\cdot(\ln n)}$.

Apply the harmonic trick again: cut the sequence into strips, strip number m will contain n's between 2^m+1 and 2m+1.

The elements in strip number m are, up to a bounded factor $\frac{1}{2^m\cdot m}$, this strip has 2^m elements, so the sum in the strip is, up to a bounded factor $\frac{1}{m}$.

So the sum over all strips behaves like $\sum_{m=?}^{\infty}\frac{1}{m}$, and that one diverges (the easiest proof for that is applying harmonic trick again.

b) As in a), apply the harmonic trick twice. Each time you get the series with the same convergence properties. After firs time you get $\sum_{n=?}^{\infty}\frac{1}{n\cdot(\ln n)^2}$, and after second time you get $\sum_{m=?}^{\infty}\frac{1}{m^2}$, and that one converges.

3. Let $\{a_n\}$ be a sequence defined by $a_0 = 1$ and recursive formula

$$a_{n+1}=\frac{1}{n+1}\sum_{k=0}^{n}\frac{a_k}{n-k+2}$$

Find the limit $\sum_{k=0}^{\infty}\frac{a_k}{2^k}$.

Solution. Consider a generating function of that sequence $f(x)=\sum_{n=0}^{\infty}a_n x^n$.

Then the limit of the series that we should compute is simply $f\left(\frac{1}{2}\right)$.

The equation $(n+1)a_{n+1}=\sum_{k=0}^{n}\frac{a_k}{n-k+2}$ after we multiply both sides by xn and sum them up turn to $\sum_{n=0}^{\infty}(n+1)a_{n+1}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}a_k x^k\frac{x^{n-k}}{n-k+2}=\sum_{m=0}^{\infty}\frac{x^m}{m+2}\sum_{k=0}^{\infty}a_k x^k$

$$f'(x) = f(x)\sum_{m=0}^{\infty}\frac{x^m}{m+2}$$

So, we should try to compute $g(x) = \sum_{m=0}^{\infty}\frac{x^m}{m+2} = \frac{1}{2} + \frac{x}{3} + \frac{x^2}{4} + \frac{x^3}{5} + \ldots.$

But we know that $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \ldots$

Therefore $g(x) = -\frac{x + \ln(1-x)}{x^2}$.

Hence $\frac{df}{f} = -\frac{x + \ln(1-x)}{x^2}dx$

$$\ln f + const = -\int\frac{dx}{x} - \int\frac{\ln(1-x)}{x^2}dx = -\ln x + \int\left(\frac{\ln(1-x)}{x}\right)' + \frac{1}{x(1-x)}dx =$$

$$= -\ln x + \frac{\ln(1-x)}{x} + \int\frac{1}{x(1-x)}dx = -\ln x + \frac{\ln(1-x)}{x} + \int\left(\frac{1}{x} + \frac{1}{1-x}\right)dx =$$

$$= -\ln x + \frac{\ln(1-x)}{x} + \ln x - \ln(1-x) = \frac{1-x}{x}\ln(1-x)$$

When $x = 0$ the right hand side might be seen as a limit (since it comes from power series) so it is -1 ; $f(0) = 1$ and its ln is 0, hence left hand side is equal to *const*, so $const = -1$.

So $f(x) = \exp\left(1 + \frac{1-x}{x}\ln(1-x)\right)$

Therefore $f\left(\frac{1}{2}\right) = \exp\left(1 + \ln\left(\frac{1}{2}\right)\right) = e^{1-\ln 2} = \frac{e}{2}$

4*. Compute $\lim_{x\to 0}\frac{\sin(\tan(x)) - \tan(\sin(x))}{\arcsin(\arctan(x)) - \arctan(\arcsin(x))}$.

Simple lemma. Suppose we have a function $f(x)$ such that $f(0) = 0$ and f has a continuous derivative around 0.

Suppose also that $u, v \to 0$ and always $u \neq v$. Then $\frac{f(u) - f(u)}{u - v} \to f'(0)$.

Proof. By Lagrange theorem, $\frac{f(u) - f(u)}{u - v} = f'(w)$ where w is between u and v, so when $u, v \to 0$, w also tends to 0. QED of lemma.

Notice that all functions are analytic.
$\sin(\tan(x)) \neq \tan(\sin(x))$ at least sometimes, for example when $x = \arctan\left(\frac{3}{2}\pi\right)$ because, then LHS is negative and RHS is positive.
Since functions are analytic, their points of coincidence are isolated, so at some neighborhood of 0, except for 0 itself, $\sin(\tan(x)) \neq \tan(\sin(x))$, so we learn from complex functions. So, at some neighborhood of 0 their inverse functions, $\arcsin(\arctan(x)), \arctan(\arcsin(x))$ also don't coincide, hence we can consider that fraction without problems.

From now on we shall say p(x) ~ q(x) to indicate that $\lim_{x\to 0}\frac{p(x)}{q(x)} = 1$.

It is known that x ~ sin x ~ tan x.
Denote $y = \tan(\sin(x))$ then y ~ x.

$$\lim_{x\to 0}\frac{\sin(\tan(x)) - \tan(\sin(x))}{\arcsin(\arctan(x)) - \arctan(\arcsin(x))} =$$

$$= \lim_{x\to 0}\frac{\sin(\tan(\arcsin(\arctan(y)))) - y}{\arcsin(\arctan(x)) - \arctan(\arcsin(x))} =$$

$$= \lim_{x\to 0}\frac{\sin(\tan(\arcsin(\arctan(y)))) - y}{x - \tan(\sin(\arctan(\arcsin(x))))}$$

The last equality follows from the above lemma, when you apply function $f(x) = \tan(\sin(x))$ to $u = \arcsin(\arctan(x)), v = \arctan(\arcsin(x))$, you get $\arcsin(\arctan(x)) - \arctan(\arcsin(x)) \sim x - \tan(\sin(\arctan(\arcsin(x))))$.

If we denote $g(y) = \sin(\tan(\arcsin(\arctan(y))))$, and its inverse function $g^{-1}(x) = \tan(\sin(\arctan(\arcsin(x))))$, then what we got is

$$\lim_{x\to 0}\frac{g(y) - y}{x - g^{-1}(x)}$$

From x ~ sin x ~ tan x it follows that $x \sim g(x) \sim g^{-1}(x)$.
But we have seen before that both numerator and denominator are nonzero, so g has no stable points in the neighborhood of 0 other than 0.

Hence Taylor series of g has at least 2 nonzero terms, and the first is x:
$g(x) = x + ax^n + \ldots$, where a is an nonzero number which I know nothing about, and n > 1 is a number which I know almost nothing about, and I don't want to (though they can be computed). Then $g^{-1}(x) = x - ax^n + \ldots$.
Now since $y \sim x$,

$$\lim_{x\to 0} \frac{g(y) - y}{x - g^{-1}(x)} = \lim_{x\to 0} \frac{ay^n + o(y^n)}{-(-ay^n + o(y^n))} = 1$$

So, the answer is 1.

5*. $\{a_n\}$ is a sequence of positive real numbers such that $a_n + a_m \geq a_{n+m}$ for all m, n.
Prove that the sequence a_n/n converges.

Proof. Firstly, $a_n \leq na_1$ hence the sequence a_n/n is bounded from above. It is also bounded from below by 0. So it is bounded. Hence it has liminf (lower limit) which will be denoted by L.
For each $\varepsilon > 0$, we can find an index k such that $a_k/k < L + \varepsilon$.
Then $a_k < k(L + \varepsilon)$.
Then $a_{mk} \leq ma_k < mk(L + \varepsilon)$.
For each natural N, we can write $N = mk + r$, where $r < k$ (division with remainder). Hence $a_N \leq a_{mk} + a_r \leq ma_k + ra_1 < mk(L + \varepsilon) + ma_1$.
Therefore $a_N/N < L + \varepsilon + ma_1/N < L + 2\varepsilon$, for sufficiently large N.
So, for any $\varepsilon > 0$, $a_N/N < L + 2\varepsilon$, for sufficiently large N.
Since L is liminf, then for sufficiently large N also $a_N/N > L - 2\varepsilon$.
Hence the sequence converges to L.