Targil 12 – some inequalities.

1. a) What is greater: e^{π} or π^{e} ?
b) The sum of several natural numbers is 2008.
What is largest possible value of their product?

a) First solution. e^{x} is a strictly convex function, so it is above the tangent line at 1. Tangent line at 1 is ex. Hence $e^{x} \geq ex$, and since convexity is strict the equality may happen only when $x = 1$.
Hence $e^{\pi/e} > e \cdot \pi/e = \pi$.
So $e^{\pi} > \pi^{e}$.

Second solution. The question is, actually, what is greater, $\sqrt[e]{e}$ or $\sqrt[\pi]{\pi}$.
To find compute the derivative of $\sqrt[x]{x}$ function (or, if you don't like the hard work, the derivative of it logarithm, $x \ln x$). You will see that the derivative is positive before e, negative after e, and 0 at e.
So $\sqrt[e]{e}$ is the greatest of all $\sqrt[x]{x}$.

b) If k>4, then k(k–2) is bigger then k, so there should be no numbers bigger than 4. If k = 4, we can replace it by 2·2 and still have the same product. Hence WLOG, we can assume we have only 2's and 3's. But 3·3 is greater than 2·2·2, so we have less than three 2's.
2004 is divisible by 3, so we can have two 2's and 668 times 3, or one 4 and 668 times 3. The product is $3^{668} \cdot 4$.

Remark. The question is, morally, which numbers, k's or m's, are more useful, which is bigger, m^{k} or k^{m}, ore in other words, $\sqrt[k]{k}$ or $\sqrt[m]{m}$. The answer is, that the best number is e, but as long as we are bound to use integer numbers, the best is 3, the next two are 2 and 4, which are equivalent since $\sqrt[2]{2} = \sqrt[4]{4}$ and other number are worse since they are worse approximations of e. If the question would be about real, and not integer numbers, we would get equal numbers (by AM-GM) very close to e.

2. Let $0 < x < \pi/2$. What is greater: $\tan(\sin(x))$ or $\sin(\tan(x))$?

First solution. Let us start with a guess. On the domain sin goes upwards from 0 to 1 so tan(sin) goes monotonically upwards. While tan(x) goes upwards from 0 to infinity hence sin(tan) oscillates on the domain, dives to negative and returns to positive many times. So, at least sometimes the first

function is bigger, and there wouldn't be a point in this question unless it is always bigger. So, we shall prove that $\tan(\sin(x)) > \sin(\tan(x))$.
It is enough to prove the claim on the domain where both functions are still growing, i.e. when $\tan(x) \le \pi/4$. At $\arctan(\pi/4)$ the $\sin(\tan(x))$ reaches its global maximum, 1, while $\tan(\sin(x))$ keeps growing.

Denote $y = \tan(\sin(x))$. Then $\sin(x) = \arctan(y)$ $^{(*)}$.
Hence the main claim is $\arcsin(y) > \tan(x)$.
It is enough to prove that the rate of growing of LHS is higher, i. e.

$$\frac{dx}{\cos^2 x} < \frac{dy}{\sqrt{1-y^2}}$$

$$1 < \frac{\cos^2 x}{\sqrt{1-y^2}} \frac{dy}{dx}$$

While from $^{(*)}$ we have

$$\cos x \cdot dx = \frac{dy}{1+y^2}$$

Hence we can substitute $\frac{dy}{dx}$ into the claim:

$$1 < \cos^3 x \frac{1+y^2}{\sqrt{1-y^2}}$$

But from $^{(*)}$ we can express $\cos x = \sqrt{1-\arctan^2 y}$ so now we have to prove arrive to an innocently looking inequality, with no composed functions:

$$1 < \left(1-\arctan^2 y\right)^{3/2} \frac{1+y^2}{\sqrt{1-y^2}}$$

Where $0 < y < 1$. Here we have square roots twice, so let's take the square:

$$\frac{1-y^2}{\left(1+y^2\right)^2} < \left(1-\arctan^2 y\right)^3$$

But we can write $\frac{1}{1+y^2} = 1 - \frac{y^2}{1+y^2}$

Hence if $z = \arctan y$, then we can rewrite the claim as

$$\left(1-\tan^2 z\right)\left(1-\sin^2 z\right)\left(1-\sin^2 z\right) < \left(1-z^2\right)^3$$

$$\left(\frac{\cos^2 z - \sin^2 z}{\cos^2 z}\right)\left(\cos^2 z\right)^2 < \left(1-z^2\right)^3$$

$$\cos(2z)\cos^2 z < \left(1-z^2\right)^3$$

$$\frac{\cos(2z)+\cos^2(2z)}{2} = \cos(2z)\frac{1+\cos(2z)}{2} < \left(1-z^2\right)^3$$

$$\frac{\cos(4z)+2\cos(2z)+1}{4} < \left(1-z^2\right)^3$$

Here $0 < z < \arctan(1) = 1/\sqrt{2} < 1$.
So, we shall estimate the left hand side by its Taylor series. We know from Lagrange remainder that if we take 4k+1 terms we shall get upper bound for cosine. Hence:

$$\frac{\cos(4z)+2\cos(2z)+1}{4} < \frac{1}{4}\left(1+1-\frac{(4z)^2}{2}+\frac{(4z)^4}{24}-\frac{(4z)^6}{6!}+\frac{(4z)^8}{8!}\right)+$$

$$+\frac{1}{2}\left(1-\frac{(2z)^2}{2}+\frac{(2z)^4}{24}-\frac{(2z)^6}{6!}+\frac{(2z)^8}{8!}\right)=$$

$$=1-3z^2+\frac{4^3+2^3}{24}z^4-\frac{4^5+2^5}{6!}z^6+\frac{4^7+2^7}{8!}z^8=1-3z^2+3z^4-\frac{22}{15}z^6+\frac{43}{105}z^8$$

So, to achieve happiness, i. e. to prove that LHS < $(1 - z^2)^3$ it is enough to prove that

$$1-3z^2+3z^4-\frac{22}{15}z^6+\frac{43}{105}z^8 < 1-3z^2+3z^4-z^6$$

$$\frac{43}{105}z^8 < \frac{7}{15}z^6$$

$$z^2 < \frac{735}{645}$$

It is true in the domain which we consider. QED.
Sorry people, less Taylor terms are simply not enough. And yes, I did the computations by hands.

Second solution (the official one). Let $f(x) = \tan(\sin(x)) - \sin(\tan(x))$. Then

$$f'(x) = \frac{\cos x}{\cos^2(\sin x)} - \frac{\cos(\tan(x))}{\cos^2 x} = \frac{\cos^3 x - \cos^2(\sin x)\cos(\tan(x))}{\cos^2(\sin x)\cos^2 x}$$

Let $0 < x < \arctan(\pi/2)$. Since cos is concave (sad) on $(0\ ,\ \pi/2)$ we get

$$\sqrt[3]{\cos^2(\sin x)\cos(\tan(x))} < \frac{\cos(\tan x)+2\cos(\sin x)}{3} \le \cos\frac{\tan x+2\sin x}{3} < \cos(x)$$

The last inequality follows from $\frac{\tan x + 2\sin x}{3} > x$, which is because

$$\left[\frac{\tan x + 2\sin x}{3}\right]' = \frac{1}{3}\left[\frac{1}{\cos^2 x} + 2\cos x\right] > \sqrt[3]{\frac{1}{\cos^2 x}\cdot\cos x\cdot\cos x} = 1$$

3. We are given 5 positive real numbers: a, b, c, d, e such that

$$a^2 + b^2 + c^2 = d^2 + e^2$$
$$a^4 + b^4 + c^4 = d^4 + e^4$$

What is greater: $a^3 + b^3 + c^3$ or $d^3 + e^3$?

First solution. Let $A=a^2$, $B=b^2$, $C=c^2$, $D=d^2$, $E=e^2$.
$P=A+B+C = D+E$, $Q = A^2+B^2+C^2 = D^2 + E^2$.
Consider in the $\{(x, y, z)\}$ space the set defined by

$$x + y + z = P$$
$$x^2 + y^2 + z^2 = Q$$

The first equation is a plane, the second is a sphere. So, the intersection is a circle. It cuts coordinate planes at points $(D, E, 0)$, $(E, D, 0)$, $(D, 0, E)$, etc… Hence it has three symmetric arcs in the positive domain $x, y, z > 0$ (one arc is where x is the greatest; another where y is greatest; the third is where z is greatest). So, let us assume that $A > B > C$ and $D > E$ so that points (A, B, C) and $(D, E, 0)$ will be one the same arc of the circle (actually, on the same half of the same arc, because $y > z$ in both cases).
Let $(X(t)\ ,\ Y(t)\ ,\ Z(t))$ be a parametric curve, going along that arc from the point $(D, E, 0)$ to the point (A, B, C), which means, for instance, that

$$(X(0)\ ,\ Y(0)\ ,\ Z(0)) = (D, E, 0)$$
$$(X(1)\ ,\ Y(1)\ ,\ Z(1)) = (A, B, C)$$

We shall also assume that the curve goes always forward along the arc with constant speed v. By upper dot we denote, as usual, derivative over t.
So, since $X + Y + Z$ and $X^2 + Y^2 + Z^2$ remain constant during the motion

$$\dot{X} + \dot{Y} + \dot{Z} = 0$$
$$2X\dot{X} + 2Y\dot{Y} + 2Z\dot{Z} = 0$$

So the vector $(\dot{X},\dot{Y},\dot{Z})$ is orthogonal to both (1, 1, 1) and (X, Y, Z) and so it is proportional to their vector product $(Y-Z, Z-X, X-Y)$.
But we know that Z is growing along the way, and X is the greatest of 3, so the coefficient of proportion is positive.
So, we have computed $(\dot{X},\dot{Y},\dot{Z})$ up to a positive coefficient.
The question which really bothers us is whether the value of the function

$$F(t) = X^{3/2} + Y^{3/2} + Z^{3/2}$$

is greater at the beginning or at the end of the motion?
Let us derive it and find its sign.

$$\dot{F}(t)=\frac{3}{2}\left(\dot{X}\sqrt{X}+\dot{Y}\sqrt{Y}+\dot{Z}\sqrt{Z}\right)=a\left((Y-Z)\sqrt{X}+(Z-X)\sqrt{Y}+(X-Y)\sqrt{Z}\right)$$

Here a is a positive coefficient so it doesn't influence the sign.
Denote $x=\sqrt{X}$, $y=\sqrt{Y}$, $z=\sqrt{Z}$. Then, since $X > Y > Z > 0$ on our half-arc, so also $x > y > z > 0$. Then

$$(Y-Z)\sqrt{X}+(Z-X)\sqrt{Y}+(X-Y)\sqrt{Z}=\left(y^2-z^2\right)x+\left(z^2-x^2\right)y+\left(x^2-y^2\right)z=$$

$$=y^2x-x^2y-z^2(x-y)+(x-y)(x+y)z=(x-y)\left(-xy-z^2+(x+y)z\right)=$$

$$=-(x-y)(x-z)(y-z)<0$$

(Of course, this expression was Vandermonde of size 3).
So, $\dot{F}(t)<0$, and the value if

$$d^3 + e^3 = \mathrm{F}(0) > \mathrm{F}(1) = a^3 + b^3 + c^3.$$

Second solution (the official). WLOG, we may assume that $a \geq b \geq c$ and $d \geq e$.
Let $c^2 = e^2 + \Delta$. Then $d^2 = a^2 + b^2 + \Delta$.
Hence $a^4+b^4+\left(e^2+\Delta\right)^4=\left(a^2+b^2+\Delta\right)^2+e^4$

$$2e^2\Delta=2a^2b^2+2\left(a^2+b^2\right)\Delta$$

$$\Delta=-\frac{a^2b^2}{a^2+b^2-e^2}$$

But $a^2+b^2-e^2\geq\frac{2}{3}\left(a^2+b^2+c^2\right)-\frac{1}{2}\left(d^2+e^2\right)=\frac{1}{6}\left(d^2+e^2\right)>0$
So Δ is negative.

$$0<c^2=e^2-\frac{a^2b^2}{a^2+b^2-e^2}=\frac{a^2e^2+b^2e^2-e^4-a^2b^2}{a^2+b^2-e^2}=\frac{\left(a^2-e^2\right)\left(e^2-b^2\right)}{a^2+b^2-e^2}$$

So $a > e > b$.
Therefore $d^2=a^2+b^2-\frac{a^2b^2}{a^2+b^2-e^2}<a^2+b^2-\frac{a^2b^2}{a^2}=a^2$.
Hence $a > d \geq e > b \geq c$.
Consider the function

$$f(x) = a^x + b^x + c^x - d^x - e^x$$

WLOG $a = 1$(if not, divide everything by a^x).
We shall prove that this function has only 2 zeroes on the real line, 2 and 4, and that f changes its sign at those points. Suppose the contrary. Then, by

Rolle's theorem, there f' has at least two distinct zeroes, $x_1 < x_2$,
$f'(x_1) = f'(x_2) = 0$.

$$f'(x) = \ln b \cdot b^x + \ln c \cdot c^x - \ln d \cdot d^x - \ln e \cdot e^x$$

Hence $\ln b \cdot b^{x_i} + \ln c \cdot c^{x_i} = \ln d \cdot d^{x_i} + \ln e \cdot e^{x_i}$ for $i = 1, 2$.
But $1 > d \geq e > b \geq c$. So

$$\frac{(-\ln b)b^{x_1} + (-\ln c)c^{x_1}}{(-\ln b)b^{x_2} + (-\ln c)c^{x_2}} \leq b^{x_1 - x_2} < e^{x_1 - x_2} \leq \frac{(-\ln d)d^{x_1} + (-\ln e)e^{x_1}}{(-\ln d)d^{x_2} + (-\ln e)e^{x_2}},$$

a contradiction.
Hence $f(x)$ has constant sign at intervals $(-\infty, 2), (2,4), (4,\infty)$. It is positive at 0, so it is negative at 3.

4. a) Let $\{x_i\}$ be a decreasing sequence of positive numbers.
Prove that $\sqrt{\sum_{i=1}^{n} x_i^2} \leq \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}}$

b) Prove that there exists a universal constant C, such that for each decreasing sequence of positive numbers

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sqrt{\sum_{i=m}^{\infty} x_i^2} \leq C \sum_{i=1}^{n} x_i$$

Solution. a) Take the square: $\sum_{i=1}^{n} x_i^2 \leq \left(\sum_{i=1}^{n} \frac{x_i}{\sqrt{i}} \right)^2$
But if You replace all numbers by lower numbers with higher indices:

$$\left(\sum_{i=1}^{n} \frac{x_i}{\sqrt{i}} \right)^2 = x_1^2 + \frac{x_2}{\sqrt{2}} \left(2x_1 + \frac{1}{\sqrt{2}} x_2 \right) + \frac{x_3}{\sqrt{3}} \left(2x_1 + 2\frac{x_2}{\sqrt{2}} + \frac{x_3}{\sqrt{3}} \right) + \ldots >$$

$$> x_1^2 + \frac{x_2^2}{\sqrt{2}} \left(2 + \frac{1}{\sqrt{2}} \right) + \frac{x_3^2}{\sqrt{3}} \left(2 + 2\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \right) + \frac{x_4^2}{\sqrt{4}} \left(2 + 2\frac{1}{\sqrt{2}} + 2\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} \right) + \ldots$$

It remains to estimate sum of inverse roots. It is easy to guess (for example, if you approximate series by integral) that it is approximately $\sqrt{n}$.
To make precise statement consider

$$\sqrt{n}-\sqrt{n-1}=\frac{\left(\sqrt{n}-\sqrt{n-1}\right)\left(\sqrt{n}+\sqrt{n-1}\right)}{\sqrt{n}+\sqrt{n-1}}=\frac{1}{\sqrt{n}+\sqrt{n-1}}<\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n-1}}\right)\Big/4$$

(Because always $\frac{1}{A+B}\le\left(\frac{1}{A}+\frac{1}{B}\right)\Big/4=\frac{A+B}{4AB}$.

Since $4AB\le(A+B)^2$, which is same as $0\le(A-B)^2$.)

Sum up such inequalities from 2 to *m* and You get

$$\sqrt{m}-1<\frac{1}{4}\left(\frac{1}{\sqrt{1}}+\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{3}}+...+\frac{2}{\sqrt{m-1}}+\frac{1}{\sqrt{m}}\right)$$

So $\sqrt{m}<\frac{2}{\sqrt{1}}+\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{3}}+...+\frac{2}{\sqrt{m-1}}+\frac{1}{\sqrt{m}}$

(The last formula could be obtained with less subtle technique, but technique is what we want to learn). Now we can finish the proof:

$$\left(\sum_{i=1}^{n}\frac{x_i}{\sqrt{i}}\right)^2>x_1^2+\frac{x_2^2}{\sqrt{2}}\left(2+\frac{1}{\sqrt{2}}\right)+\frac{x_3^2}{\sqrt{3}}\left(2+2\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\right)+...>x_1^2+x_2^2+x_3^2+x_4^2...$$

b)

$$\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\sqrt{\sum_{i=m}^{\infty}x_i^2}\le\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\sum_{i=m}^{\infty}\frac{x_i}{\sqrt{i+1-m}}=\sum_{i=1}^{\infty}x_i\sum_{m=1}^{i}\frac{1}{\sqrt{m}}\cdot\frac{1}{\sqrt{i+1-m}}$$

All we have to do is to estimate

$$\sum_{m=1}^{i}\frac{1}{\sqrt{m}}\cdot\frac{1}{\sqrt{i+1-m}}=\frac{1}{i}\sum_{m=1}^{i}\frac{1}{\sqrt{\frac{m}{i}}}\cdot\frac{1}{\sqrt{1+\frac{1}{i}-\frac{m}{i}}}\sim\int_0^1\frac{dx}{\sqrt{x}\sqrt{1-x}}$$

Because it is approximately Riemann's sum for that integral. Actually, if we divide the interval [0,1] into *i* equal parts, then in the part number *m* we have $x<\frac{m}{i},1-x<1+\frac{1}{i}-\frac{m}{i}$ hence $\frac{1}{\sqrt{\frac{m}{i}}}\cdot\frac{1}{\sqrt{1+\frac{1}{i}-\frac{m}{i}}}<\frac{1}{\sqrt{x}\sqrt{1-x}}$ and

$$\sum_{m=1}^{i}\frac{1}{\sqrt{m}}\cdot\frac{1}{\sqrt{i+1-m}}=\frac{1}{i}\sum_{m=1}^{i}\frac{1}{\sqrt{\frac{m}{i}}}\cdot\frac{1}{\sqrt{1+\frac{1}{i}-\frac{m}{i}}}<\int_0^1\frac{dx}{\sqrt{x}\sqrt{1-x}}$$

So, it remains to compute that integral, and it will serve as (a pretty tight) universal constant. It is done by a trigo substitution: $x = \sin^2 t$

$dx = 2\sin t\cos t\,dt$

$$\int_0^1 \frac{dx}{\sqrt{x}\sqrt{1-x}} = \int_0^{\pi/2} 2dt = \pi.$$

There is a more elementary way to find a less tight bound:

$$\sum_{m=1}^{i} \frac{1}{\sqrt{m}} \cdot \frac{1}{\sqrt{i+1-m}} = 2\sum_{m=1}^{i/2} \frac{1}{\sqrt{m}} \cdot \frac{1}{\sqrt{i+1-m}} \le 2\frac{1}{\sqrt{i/2}} \sum_{m=1}^{i/2} \frac{1}{\sqrt{m}} \le 2\frac{1}{\sqrt{i/2}} 2\sqrt{i/2} = 4$$

5.** Let $\{a_n\}$ be a sequence of positive numbers, such that $\sum_{n=1}^{\infty} a_n < \infty$.

***a*)** Prove that $\sum_{n=1}^{\infty} \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} < e\sum_{n=1}^{\infty} a_n$.

***b*)** Prove that the constant in the inequality can't be improved, i. e. for any positive ε we can find a sequence of positive numbers s.t. $\sum_{n=1}^{\infty} a_n < \infty$ and

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} > (e-\varepsilon)\sum_{n=1}^{\infty} a_n$$

This inequality is called ***Carleman inequality***. Carleman is a famous Swedish mathematician.
This inequality was given at IMC and it was the most expensive problem in its year. We shall start with b) part, since it is easier and gives a clue to a).

b) Consider the sequence $a_n = \frac{1}{n^u}$ where *u* is only slightly bigger then 1.
Then the sum of a_n is bounded, but very big, so the first few terms are only a small part of the sum. For large *n*'s,

$$\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} = \left(\frac{1}{\sqrt[n]{1 \cdot 2 \cdot \ldots \cdot n}}\right)^u \sim \left(\frac{e}{n}\right)^u = e^u a_n \sim e a_n$$

Hence for *u* sufficiently close to 1, the LHS and RHS of Carleman inequality are approximately equal. QED.

This gives a clue to *a* part. It is very natural to try and decompose Carleman inequality into sum of Cauchy (AM-GM) inequalities, say

$$\sqrt[n]{b_{n,1} \cdot b_{n,2} \cdot \ldots \cdot b_{n,n}} \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} \le \frac{b_{n,1}a_1 + b_{n,2}a_2 + \ldots + b_{n,n}a_n}{n}$$

Then, if you choose the coefficients smartly, so that $\sqrt[n]{b_{n,1}\cdot b_{n,2}\cdot\ldots\cdot b_{n,n}}$ is less than e times sum of all coefficients before each a_k, you will get Carleman. But how to guess those coefficients?
It is well known that Cauchy inequality becomes equality only if all the numbers in it are the same. Assume (though it is literally wrong, but morally almost true), that Carleman inequality becomes equality for the sequence $a_n = \frac{1}{n}$. So, the coefficients should be chosen so, that the numbers will become equal, so a_n should be multiplied by n before applying Cauchy. That was the main philosophical idea, now we shall execute it.

a) We shall start with reminding classical calculus lemmas:

Lemma 0. $(1+a)^n \geq 1 + na$ if $1+a > 0$ and n is a natural number. (Bernoulli inequality). Equality holds only if $a = 0$ or $n = 1$.

Lemma 1. $b_n = \left(1+\frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n < e$.

Lemma 2. $\frac{n+1}{e} < \sqrt[n]{n!}$.

Proof of lemma 0. By induction, for n = 1 it is obvious, for n+1, supposing it was proven for n, it follows:
$(1+a)^{n+1} = (1+a)^n(1+a) \geq (1 + na)(1+a) = 1 + (n+1)a + na^2 \geq 1 + (n+1)a$

Proof of lemma 1. It is well known that $b_n \to e$. So, it would be enough to prove that b_n is monotonically increasing, i. e. $b_n < b_{n+1}$.

$$\left(\frac{n+1}{n}\right)^n < \left(\frac{n+2}{n+1}\right)^{n+1}$$

$$\frac{n}{(n+1)} < \left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1}$$

$$1-\frac{1}{n+1} < \left(1-\frac{1}{(n+1)^2}\right)^{n+1}$$

That is a direct consequence of Bernoulli inequality (lemma 0).

Proof of lemma 2. Since $b_1 < b_2 < \ldots < b_n < e$ we get

$$\frac{n+1}{\sqrt[n]{n!}} = \sqrt[n]{\frac{(n+1)^n}{n!}} = \sqrt[n]{b_1 \cdot b_2 \cdot \ldots \cdot b_n} < e$$

Finally, let's prove Carleman inequality. By Cauchy and lemma 2:

$$\frac{n+1}{e}\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} < \sqrt[n]{n! \cdot a_1 \cdot a_2 \cdot \ldots \cdot a_n} \le \frac{1 \cdot a_1 + 2 \cdot a_2 + 3 \cdot a_3 + \ldots + n \cdot a_n}{n}$$

Hence $\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} < e\frac{1 \cdot a_1 + 2 \cdot a_2 + 3 \cdot a_3 + \ldots + n \cdot a_n}{n(n+1)}$.

Sum all these inequalities. You get

$$\sum_{n=1}^{\infty}\sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} < e\sum_{k=1}^{\infty}a_k \sum_{n=k}^{\infty}\frac{k}{n(n+1)} = e\sum_{k=1}^{\infty}ka_k \sum_{n=k}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = e\sum_{k=1}^{\infty}ka_k \frac{1}{k} = e\sum_{k=1}^{\infty}a_k$$

QED.