Targil 12 – some inequalities.

1. a) What is greater: e^{π} or π^{e} ?
b) The sum of several natural numbers is 2008.
What is largest possible value of their product?

2. Let $0 < x < \pi/2$. What is greater: $\mathrm{tg}(\sin(x))$ or $\sin(\mathrm{tg}(x))$?

3. We are given 5 positive real numbers: a, b, c, d, e such that

$$a^2 + b^2 + c^2 = d^2 + e^2$$
$$a^4 + b^4 + c^4 = d^4 + e^4$$

What is greater: $a^3 + b^3 + c^3$ or $d^3 + e^3$?

4. a) Let $\{x_i\}$ be a decreasing sequence of positive numbers.

Prove that $\sqrt{\sum_{i=1}^{n} x_i^2} \le \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}}$

b) Prove that there exists a universal constant C, such that for each decreasing sequence of positive numbers

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sqrt{\sum_{i=m}^{\infty} x_i^2} \le C \sum_{i=1}^{n} x_i$$

5**. Let $\{a_n\}$ be a sequence of positive numbers, such that $\sum_{n=1}^{\infty} a_n < \infty$.

a) Prove that $\sum_{n=1}^{\infty} \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} < e \sum_{n=1}^{\infty} a_n$.

b) Prove that the constant in the inequality can't be improved, i. e. for any positive ε we can find a sequence of positive numbers s.t. $\sum_{n=1}^{\infty} a_n < \infty$ and

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n} > (e - \varepsilon) \sum_{n=1}^{\infty} a_n$$