Targil 11.
Periodicity and sequences.

1*. A necklace consists of R red and B blue beads. We say that it is good, if for any two substring of the same length a number of red beads in them differ by 1 at most.
a. Prove that a good necklace exists for each R and B.
b. Prove that it is unique up to rotation.

Solution. Suppose that R ≥ B. Then between each two blue beads there is at least one red bead. If not, and there are two adjacent blue beads, then there also are two adjacent red beads, so there are two substrings of 2 which differ by 2.
So, each necklace of the kind we consider has substrings of red, delimited by single blue beads. For each such necklace we can consider a reduced necklace of R – B red beads and B blue beads, by taking one red bead out of each red substring of that kind.
The original necklace can be reconstructed easily from the reduced one, by inserting one red bead between each pair of blue beads.

Lemma. The reduced necklace is good iff the original necklace is good.

The lemma is easily proved. If the reduced necklace is bad, there are two substrings of equal length, one having precisely k, another having at least k+2 blue beads. By taking minimal substrings of equal length which demonstrate badness of the reduces necklace, we can replace condition "at least k+2" by "precisely k+2". Now enhance it back, we have to add red bead to each red substring, and we can add red bead on both ends of substring or we can not, as we wish. So in the enhanced necklace we also have two substrings of the same length, one with k and another with k+2 blue beads.
The other direction is even easier, we have two different substrings, one with only k and another with at least k+2 blue beads. Now we shrink it, and the second substring shrinks more than the first, so we can append new beads to it, to keep it as long as the first substring. But the first substring has only k blue beads, and the second has at least k+2 and might get some new ones. QED of the lemma.

Hence, to construct a good circle for R and B, first construct a good circle for R – B and B, it is already done by induction (base of induction is obvious), and then enhance it.
To prove that two different good circles are actually the same, shrink both, use induction, and enhance them back again.

Remark. I didn't formulate things accurately here, but there are two ways to formulate proofs like that. First way: prove it for n = 1, and then show that if it is true for n < N then it is also true for n = N.
The second claim: choose minimal n for which the claim is wrong, and create a contradiction. The second way of proving such claims is shorter when You write it down.

2. A sequence of natural numbers is defined by recursive formula $a_{n+1} = a_0^{a_n}$. Show that the a_n stabilizes modulo m, for each natural number m.
("Stabilizes" means that we start getting always the same number after some index).

Solution. Consider the minimal number m for which there is such a_0 so that the claim is wrong. If m is decomposable into product of two relatively prime numbers, p and q, then the sequence will stabilize mod p and mod q so by the Chinese Remainder Theorem, it will stabilize mod m, which is impossible. Hence m is a power of a prime, $m = p^k$.
If a_0 is divisible by p, than its large powers are divisible by m, so it is not the case.
So, a_0 and m are co-prime. Hence a_0^n mod m depend only on n mod $\varphi(m)$.
But a_n stabilizes mod $\varphi(m)$, because $\varphi(m) < m$, hence $a_0^{a_n}$ stabilizes mod m. QED.

3. (a) Winnie the Pooh and Piglet walk over the infinite street, tiled by blocks 1 feet long. They start at the first corner of the first block. They make constant strides, Pooh of length p and Piglet of length q, both p and q are irrational numbers of feet. After the first tile, each tile is stepped on by precisely one animal. (They walk on their toes.)
For which p and q can it happen? Find the precise condition.
(b) Is it possible to decompose the set of natural (integer positive) numbers into disjoint union of two strictly increasing sequences, $\{a_n\}$ and $\{b_n\}$, such that $b_n = n + a_n$, for each n ?

Solution. (a) The precise condition is $\frac{1}{p}+\frac{1}{q}=1$.
It is easy to understand why this condition is necessary. Because if we look at first N tiles, where N is large, Pooh will take $\sim\frac{N}{p}$ tiles and Piglet $\sim\frac{N}{q}$ tiles and together they should take ~ N tiles so $\frac{N}{p}+\frac{N}{q}\sim N$, when N goes to infinity, we see $\frac{1}{p}+\frac{1}{q}=1$.
Now we shall so this condition is also sufficient. Before the end of N'th tile Pooh will take precisely $\left[\frac{N}{p}\right]$ tiles and Piglet precisely $\left[\frac{N}{q}\right]$ tiles. Both $\frac{N}{p}$ and $\frac{N}{q}$ are not integer so since $\frac{N}{p}+\frac{N}{q}=N$ is integer we get $\left[\frac{N}{p}\right]+\left[\frac{N}{q}\right]=N-1$.
Hence $\left[\frac{N+1}{p}\right]+\left[\frac{N+1}{q}\right]=N$. The difference between two last expression is 1, so there is one more animal trace on N+1 first tiles than on N first tiles. So, precisely one animal stepped on tile N+1.

Remark. It is interesting what would happen for 3 animals, say Pooh, Piglet and Rabbit, or four animals, with Tigger for example. Of course $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$ or $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{t}=1$ will be necessary condition, but I am not sure whether it is sufficient.

(b) We shall see two solutions. From first solution it will be clear that this decomposition is unique, but not how it looks like. From the second it will be clear how precisely does this decomposition behave, but not the uniqueness.

First solution. Paint all natural numbers in two colors, pink and brown, in the following way. In the beginning all numbers are colorless. At step number k choose the least colorless number, call it a_k and paint it pink. Then take number $a_k + k$, call it b_k and paint it brown. It is obvious that ak is a strictly increasing sequence, hence so is b_k , so b_k is bigger than all numbers which were painted earlier, so the numbers don't repeat themselves. It is also obvious that all numbers will be painted after $\aleph_0$ steps.

Second solution. Let φ be the positive root of quadratic equation $\varphi^2 = \varphi + 1$.
Then $\frac{1}{\varphi^2}+\frac{1}{\varphi}=1$. So if the bear makes φ^2 steps, and his friend φ steps (maybe because they are superstitious and think that will bring them gold), then each tile will be stepped upon once. So, natural numbers are nicely decomposed into 2 sequences : $a_N = \lfloor N\varphi \rfloor$ and $b_N = \lfloor N\varphi^2 \rfloor$, as we have proven in (a).
But $a_N + N = \lfloor N\varphi \rfloor + N = \lfloor N(\varphi+1) \rfloor = \lfloor N\varphi^2 \rfloor = b_N$.

4. Two infinite sequences are given, $\{a_i\}$ of period n and $\{b_i\}$ of period m.
$a_k = b_k$ for $k < m+n$. Show that these two sequences coincide.

Solution. If m and n have common divisor $r > 1$, we can consider r pairs of sub-sequences $\{a_{ir+k}\}$ and $\{b_{ir+k}\}$, where k is a constant number, $0 \le k < r$.
This way we reduce the problem to the same problem for sequences with lesser periods, $m' = m/r$ and $n' = n/r$. So, it is enough to prove the statement when m and n are coprime.
For any k < m we have $b_k = a_k = a_{k+n} = b_{k+n}$.
Since b_k depends only on k mod m, we can visualize all values of b_k as a vertices of regular m-gon, and at vertex number k we write number b_k.
Consider all lines, connecting vertex number k to vertex number $k+n$ (mod m).
The vertices of regular m-gon and those lines form a graph. This graph is a circle of length m, because m and n are coprime, so making jumps by n mod m will bring you to original point only after m moves. So, this graph is connected, and it will remain connected even when we erase one edge.
As we saw before, $b_k = b_{k+n}$ for any k < m, so if we take only m–1 of those edges, the connected vertices will all have the same value, but the graph is connected, so all values of b sequence are the same, so b is constant, so a is constant at its first $m + n - 1$ elements, so a and b are equal.

5.** The set of natural numbers is decomposed into a disjoint union of arithmetic progressions. Show that some two of those progressions have the same step.

Solution. Let P be common multiples of all periods. After certain moment, the only thing which influences the belonging of a number to any of those sequences is its remainder mod P. Remainders k mod P can be represented as vertexes of the regular polygon, $e^{2ki/P}$. Remainders corresponding to one arithmetic progression are vertexes of regular sub-polygon. The sequence of largest difference correspond to the sub-polygon with fewest number of vertexes, n. Consider function z^n. The sum of this function over all vertices of the polygon is 0. The sum of this function over vertices of regular polygon with more than n vertices is obviously 0. The sum over vertices of polygon with precisely n vertices is nonzero. So there should be more than one polygon of n vertices.