Targil 11.

Periodicity and sequences.

1*. A necklace consists of R red and B blue beads. We say that it is good, if for any two substring of the same length a number of red beads in them differ by 1 at most.
a. Prove that a good necklace exists for each R and B.
b. Prove that it is unique up to rotation.

2. A sequence of natural numbers is defined by recursive formula $a_{n+1} = a_0^{a_n}$. Show that the a_n stabilizes modulo m, for each natural number m.
("Stabilizes" means that we start getting always the same number after some index).

3. (a) Winnie the Pooh and Piglet walk over the infinite street, tiled by blocks 1 feet long. They start at the first corner of the first block. They make constant strides, Pooh of length p and Piglet of length q, both p and q are irrational numbers of feet. After the first tile, each tile is stepped on by precisely one animal. (They walk on their toes.)
For which p and q can it happen? Find the precise condition.
(b) Is it possible to decompose the set of natural (integer positive) numbers into disjoint union of two strictly increasing sequences, $\{a_n\}$ and $\{b_n\}$, such that $b_n = n + a_n$,
for each n ?

4. Two infinite sequences are given, $\{a_i\}$ of period n and $\{b_i\}$ of period m.
$a_k = b_k$ for $k < m+n$. Show that these two sequences coincide.

5**. The set of natural numbers is decomposed into a disjoint union of arithmetic progressions. Show that some two of those progressions have the same step.