Targil 10.

Some linear algebgra, from former IMC mostly.

**1.** Let $n \geq 2$ be an integer. What is the minimal and the maximal possible rank of a $n \times n$ matrix whose $n^2$ entries are precisely 1, 2, 3, … , $n^2$ ?

Answer. Maximum $n$, minimum 2.

**Solution.** ***Minimum.*** Write all the numbers in their natural order. Then all the lines are linear combinations of the first two. Because the difference of first two lines is (1 1 … 1) and by adding this line to the first line you can get all lines.

You can't achieve rank 1. If You can, lets permute rows and columns so that $a_{11}=1$, $a_{11} < a_{12} < \ldots < a_{1n}$, $a_{11} < a_{21} < \ldots < a_{n1}$.

Then, since every $2 \times 2$ minor is 1, we have $a_{km} = a_{k1}\, a_{1m}$.

From this follows, that the lower-right corner number is bigger than any other matrix entry by 2 at least.

***Maximum.*** Take all entries on main diagonal to be odd, and all elements below it – even. Then det mod 2 is 1, so det is odd. So det isn't 0.

So rank is $n$.

**2.** A polynomial $P(x_1, x_2, \ldots, x_k)$ is called good if there exist $2 \times 2$ real matrices $A_1$ , $A_2$, … , $A_k$ such that $P(x_1, x_2, ..., x_k) = \det\left(\sum_{i=1}^{n} x_k A_k\right)$.

Find all values of $k$ for which all homogeneous polynomials of degree 2 of $k$ variables are good.

**Solution.** By playing a bit with numbers, it is easy to construct an example for 2:

$$\left| x\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ -c & b \end{pmatrix} \right| = \left| \begin{pmatrix} x & y \\ -cy & ax+by \end{pmatrix} \right| = ax^2 + bxy + cy^2$$

It is always easy to reduce number of variables. So for 1 it works also.

Now the question is, which polynomial of degree 3 can we construct. All possible polynomials are

$$\left| x\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + y\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} + z\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \right| =$$

$$= \left| \begin{pmatrix} a_1x + b_1y + c_1z & a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z & a_4x + b_4y + c_4z \end{pmatrix} \right| =$$

$$= (a_1x + b_1y + c_1z)(a_4x + b_4y + c_4z) - (a_2x + b_2y + c_2z)(a_3x + b_3y + c_3z)$$

But each homogenous polynomial of degree 2 in 3 variables can be seen as a bilinear form, in which you substitute twice the same vector, or

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The form $(a_1x + b_1y + c_1z)(a_4x + b_4y + c_4z)$ or $-(a_2x + b_2y + c_2z)(a_3x + b_3y + c_3z)$ corresponds to the matrix of rank 1.

So, the question is, which forms can be presented as sum of two matrixes of rank 1, and those are forms that can be presented as matrixes of rank 2. Matrices of rank 2 always have nontrivial kernels, so their forms become 0 on non-zero vectors. That is why polynomial $x^2 + y^2 + z^2$ can't be represented by matrix of rank 2, since it is never 0 on real nonzero vector.

**Remark.** Notice, that quadratic form $x^2 + y^2 + z^2$ can be represented by many different matrices, precisely by any matrix of the form $\begin{pmatrix} 1 & -a & -c \\ a & 1 & -b \\ c & b & 1 \end{pmatrix}$.

**3.** Let $A$ be a real $4\times2$ matrix and $B$ be a real $2\times4$ matrix such that $AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$. Find $BA$.

**Solution.** Matrix A can be written as $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and matrix B as ( $B_1$ $B_2$)

where $A_1$, $A_2$, $B_1$, $B_2$ are 2×2 blocks.

$AB = \begin{pmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{pmatrix}$, hence A1 is inverse to B1, and $A_2$ is inverse to $B_2$.

Hence $BA = B_1A_1 + B_2A_2 = \mathbf{1} + \mathbf{1} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

**4.** Let $AB$ be real n×n matrices such that $AB + A + B = 0$.
Prove that they commute, i. e. $AB = BA$ .

**Solution.** Add unit matrix to both sides of the given identity. You get

$$(\mathbf{1}+A)(\mathbf{1}+B) = \mathbf{1}.$$

That means, matrices $\mathbf{1}+A$ and $\mathbf{1}+B$ are inverse to each other, hence

$$(\mathbf{1}+B)(\mathbf{1}+A) = \mathbf{1}$$

So

$$BA + A + B = 0$$

$$AB = -(A + B) = BA .$$

**5.** $A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_1 & a_0 & a_1 & \dots & a_{n-1} \\ a_2 & a_1 & a_0 & \dots & a_{n-2} \\ . & . & . & \dots & . \\ a_n & a_{n-1} & a_{n-2} & \dots & a_0 \end{pmatrix}$

Compute det $A$ if

a) $a_k = a_0 + d \cdot k$

b) $a_k = a_0 \cdot q^k$

**Solution. a)** Subtract second row from the first, then third from the second and so on. After n operations You get a matrix of same determinant:

$$\begin{pmatrix} -d & d & d & \dots & d \\ -d & -d & d & \dots & d \\ -d & -d & -d & \dots & d \\ . & . & . & \dots & . \\ a_n & a_{n-1} & a_{n-2} & \dots & a_0 \end{pmatrix}$$

Now subtract row n-2 from row n-1, then row n-3 from n-2, and so on, in the end subtract the first row from the second. After n-1 operations You still get a matrix of the same determinant:

$$\begin{pmatrix} -d & d & d & \dots & d \\ 0 & -2d & 0 & \dots & 0 \\ 0 & 0 & -2d & \dots & 0 \\ . & . & . & \dots & . \\ a_n & a_{n-1} & a_{n-2} & \dots & a_0 \end{pmatrix}$$

Now, there are just 2 permutation which we need to count, since in all the rows except the first and the last You have to choose the diagonal element, hence the determinant is

$$(-2d)^{2n-1}(-da_0 - da_n) = (-2d)^{2n}(a_0 + dn/2)$$

**b)** Subtract q times second row from the first, then q times third from the second and so on. After n operations You get a matrix of same determinant:

$$\begin{pmatrix} a_0 - qa_1 & 0 & 0 & \dots & 0 \\ * & a_0 - qa_1 & 0 & \dots & 0 \\ * & * & a_0 - qa_1 & \dots & 0 \\ . & . & . & \dots & . \\ * & * & * & \dots & a_0 \end{pmatrix}$$

I didn't write the terms below the diagonal, because they don't matter. The only nonzero permutation which remains here is diagonal, so determinant is $a_0^{\,n}\,(1-q^2)^{\,n-1}$.