Targil 10.
Some linear algebgra, from former IMC mostly.

1. Let $n \geq 2$ be an integer. What is the minimal and the maximal possible rank of a $n \times n$ matrix whose n^2 entries are precisely $1, 2, 3, \ldots, n^2$?

2. A polynomial $P(x_1, x_2, \ldots, x_k)$ is called good if there exist 2×2 real matrices $A_1, A_2, \ldots, A_k$ such that $P(x_1, x_2, ..., x_k) = \det\left(\sum_{i=1}^{n} x_k A_k\right)$.
Find all values of k for which all homogeneous polynomials of degree 2 of k variables are good.

3. Let A be a real 4×2 matrix and B be a real 2×4 matrix such that

$$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$ Find BA.

4. Let AB be real $n \times n$ matrices such that $AB + A + B = 0$.
Prove that they commute, i. e. $AB = BA$.

5. $$A = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_1 & a_0 & a_1 & \ldots & a_{n-1} \\ a_2 & a_1 & a_0 & \ldots & a_{n-2} \\ . & . & . & \ldots & . \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_0 \end{pmatrix}$$

Compute det A if
a) $a_k = a_0 + d \cdot k$
b) $a_k = a_0 \cdot q^k$